Triangle Algebras: a Formal Logic Approach to Interval-Valued Residuated Lattices

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Abstract

In this paper, we introduce triangle algebras: a variety of residuated lattices equipped with approximation operators, and with a third angular point u, different from 0 and 1. We show that these algebras serve as an equational representation of interval-valued residuated lattices (IVRLs). Furthermore, we present Triangle Logic (TL), a system of many-valued logic capturing the tautologies of IVRLs. Triangle algebras are used to cast the essence of using closed intervals of $\mathcal{L}$ as truth values into a set of appropriate logical axioms. Our results constitute a crucial first step towards solving an important research challenge: the axiomatic formalization of residuated t-norm based logics on $\mathcal{L}^I$, the lattice of closed intervals of [0,1], in a similar way as was done for formal fuzzy logics on the unit interval.

Key words:
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1 Introduction and Preliminaries

1.1 Residuated Lattices

The concept of residuated lattice [23] will be used extensively in this paper. Therefore we recall its definition and summarize its main properties.

Definition 1 *A residuated lattice*[2] *is a structure* $\mathcal{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ *in which* $\sqcap, \sqcup, *$ *and* $\Rightarrow$ *are binary operators on the set* L *and*

- $(L, \sqcap, \sqcup)$ *is a bounded lattice with* 0 *as smallest and* 1 *as greatest element,*
- $*$ *is commutative and associative, with* 1 *as neutral element, and*
- $x * y \leq z$ *iff* $x \leq y \Rightarrow z$ *for all* x, y *and* z *in* L *(residuation principle).*

We will use the notations $\neg x$ *for* $x \Rightarrow 0$ *and* $x \Leftrightarrow y$ *for* $(x \Rightarrow y) \sqcap (y \Rightarrow x)$.

For further use, we give some general properties of residuated lattices. See e.g. [32] for their proofs.

Proposition 2 *Let* $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ *be a residuated lattice. Then the following properties are valid, for every* x, y *and* z *in* L*:*

(1) $x \Rightarrow y$ *is equal to the largest element* z *in* L *that satisfies* $x * z \leq y$, *so we have* $x \Rightarrow y = \sup\{z \in L \mid x * z \leq y\}$
(2) $x \leq \neg\neg x$,
(3) *if* $x \leq y$ *then* $\neg y \leq \neg x$,
(4) $x \leq y$ *iff* $x \Rightarrow y = 1$,
(5) $x = y$ *iff* $x \Leftrightarrow y = 1$,
(6) $x \leq y \Rightarrow (x * y)$,
(7) $x * (x \Rightarrow y) \leq y$ *(in particular:* $x * \neg x = 0$*),*
(8) $x * (y \sqcup z) = (x * y) \sqcup (x * z)$,
(9) $\neg(x \sqcup y) = \neg x \sqcap \neg y$,
(10) $\neg(x * y) = x \Rightarrow \neg y$.

Moreover, in residuated lattices with involutive negation ($\neg\neg x = x$),

(11) $\neg(x \sqcap y) = \neg x \sqcup \neg y$,
(12) $x \Rightarrow y = \neg y \Rightarrow \neg x$,
(13) $\neg(x \Rightarrow y) = x * \neg y$,

[2] In literature (e.g. in [23]), the name residuated lattice is sometimes used for more general structures than what we call residuated lattices. In the most general terminology, our structures would be called bounded integral commutative residuated lattices.

for every x and y in L.

Properties 9 and 11 are known as the de Morgan laws.

The best-known residuated lattices are those on the unit interval (with the usual ordering), which are called the standard residuated lattices [15]. It can easily be seen that $*$ is a left-continuous triangular norm (t-norm) on $[0,1]$ if $([0,1], \min, \max, *, \Rightarrow, 0, 1)$ is a residuated lattice. Recall that a t-norm on the unit interval is an increasing, commutative and associative binary mapping T that satisfies $T(x, 1) = x$ for all x in $[0,1]$. In general, t-norms can be defined on any bounded lattice $(L, \sqcap, \sqcup)$ (or even any bounded poset). If this lattice is complete (as is the case for the unit interval), it is possible to define the residual implicator I_T of a t-norm T: $I_T(x, y) = \sup\{z \in L \mid T(x, z) \leq y\}$ for all x and y in L. However, this does not always yield a residuated lattice. The t-norms for which we do obtain residuated lattices will be called residuated t-norms. For example, on the unit interval: $([0,1], \min, \max, T, I_T, 0, 1)$ is a residuated lattice iff T is a left-continuous t-norm on $[0,1]$.
By Proposition 2(1), the implicator $\Rightarrow$ in a residuated lattice is completely determined by the other operators: it is the residual implicator $\Rightarrow_*$ of $*$.

1.2 Formal Fuzzy Logics

Formal fuzzy logics are generalizations of classical logic that allow us to reason gradually. Indeed, in the semantics of these logics, formulas can be assigned not only 0 and 1 as truth values, but also elements of [0,1], or, more generally, of a bounded lattice $\mathcal{L}$. The partial ordering of $\mathcal{L}$ then serves to compare the truth values of formulas which can be true to some extent. The best-known examples of formal fuzzy logics are Monoidal T-norm based Logic (MTL, Esteva and Godo [14]), Basic Logic (BL, Hájek [21]), Gödel logic (G, [19]) and Łukasiewicz logic (Ł, [25]). For all of these logics, which are fully described in terms of axioms, with the modus ponens as deduction rule, soundness and completeness with respect to a corresponding class of algebraic structures can be proved. For instance, a formula can be deduced in MTL iff it is true (i.e., has truth value 1) in every prelinear residuated lattice (prelinearity means that $(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$ for all x and y in L). A prelinear residuated lattice is also called an MTL-algebra. Note that every residuated lattice on the unit interval is an MTL-algebra. A stronger result (so-called standard completeness) can be proven for MTL: a formula is deducible in MTL iff it holds in every residuated lattice $([0,1], \min, \max, *, \Rightarrow, 0, 1)$ on the unit interval. The other mentioned logics emerge by adding axioms to MTL, and are sound and complete w.r.t. subclasses of MTL-algebras, and standard complete w.r.t. specific subclasses of residuated lattices on $[0,1]$. For an overview of the state-of-the-art on formal fuzzy logics, we refer to [15], [20] and the Special Section "What is Fuzzy Logic" in Fuzzy Sets and Systems, Vol. 5(5), 2006.

1.3 Intervals as Truth Values

Research on formal fuzzy logics has centered on prelinear residuated structures; indeed, all of the above-mentioned logics presuppose prelinearity. In [1], the authors argument that prelinearity is in fact the essence of fuzzy logics. However, while this property holds in every residuated lattice $([0, 1], \min, \max, *, \Rightarrow, 0, 1)$, it is usually not preserved for closed intervals of a bounded lattice $\mathcal{L}$; for example, it was shown in [7] that no MTL-algebra exists on the lattice $\mathcal{L}^I = (L^I, \sqcap, \sqcup)$, shown graphically in Figure 1 and defined by

$$L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 \leq x_2\},$$
$$[x_1, x_2] \sqcap [y_1, y_2] = [\min(x_1, y_1), \min(x_2, y_2)],$$
$$[x_1, x_2] \sqcup [y_1, y_2] = [\max(x_1, y_1), \max(x_2, y_2)],$$

and whose partial ordering $\leq_{L^I}$ is given by componentwise extension of $\leq$,

$$[x_1, x_2] \leq_{L^I} [y_1, y_2] \iff x_1 \leq y_1 \text{ and } x_2 \leq y_2.$$

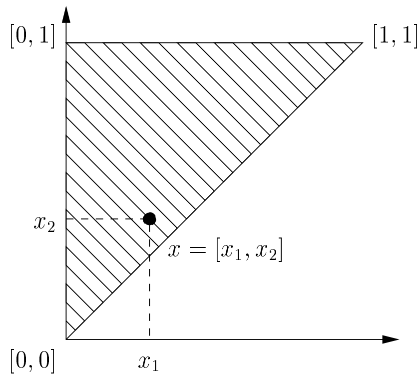


Fig. 1. The lattice $\mathcal{L}^I$

Residuated lattices can be constructed on top of $\mathcal{L}^{\mathcal{I}}$ quite easily, and as extensive research (see e.g. [6]) has pointed out, some of them rival their counterparts on [0,1] for the properties they satisfy. For example, the t-norm $\mathcal{T}$ and its residual implication $\mathcal{I}$, defined by $\mathcal{T}([x_1, x_2], [y_1, y_2]) = [\max(0, x_1 + y_1 - 1), \max(0, x_1 + y_2 - 1, x_2 + y_1 - 1)]$ and $\mathcal{I}([x_1, x_2], [y_1, y_2]) = [\min(1, 1 - x_1 + y_1, 1 - x_2 + y_2), \min(1, 1 - x_1 + y_2)]$, are continuous, the corresponding negation is an involution, $\sqcup$-definability [14,15] and the Smets-Magrez axioms [30] are satisfied, $\mathcal{T}$ is (strongly) Archimedean and nilpotent. Moreover, some important representation theorems on $[0,1]$ can be generalized to $\mathcal{L}^I$ if this t-norm is used. For a comprehensive overview, we refer to [10].

One way to interpret these intervals is to view them as approximations of truth values in $[0,1]$. For example, a truth value $[a,b]$ might mean that the exact, but unknown, truth value is definitely greater than or equal to a and smaller than or equal to b. However, in Remark 33 we will argue that this is probably not the best interpretation possible. In general, in this paper we concentrate on mathematical issues and leave the 'interpretation issues' for further papers.

1.4 Outline of the Paper

In view of the above considerations, the goal of this paper is to chart the landscape of formal fuzzy logics beyond prelinearity, and, more specifically, to develop a logic that formally characterizes tautologies (true formulas) in interval-valued residuated lattices (IVRLs[3]). In this respect, first note that by simply omitting the logical axiom corresponding to prelinearity from MTL, we arrive at Monoidal Logic (ML, Höhle [23]), whose algebraic counterpart is the complete class of residuated lattices. In other words, this logic seems too general for our purposes, and we need to extend it with suitable axioms to replace prelinearity. To achieve this, we propose the use of modal-like operators. Such operators have been studied from various angles, in formal fuzzy logics, modal logics, rough set theory, ... (see e.g. [2,4,21,27]), and serve well to capture the triangular structure (as seen also in Figure 1) of IVRLs.

This paper proceeds as follows: first, in Section 2.1, we introduce triangle algebras (residuated lattices enhanced with modal-like operators subject to a set of equation-based conditions) and show (Section 2.2) that every triangle algebra uniquely determines an IVRL, and vice versa. To illustrate the relevance of these concepts, we relate them to existing work about residuated t-norms on $\mathcal{L}^I$. We will also compare triangle algebras to related algebraic structures (Section 2.3). Next, in Section 3, the conditions of a triangle algebra are naturally transformed to logical axioms, generating Triangle Logic (TL). We show soundness and completeness of TL w.r.t. triangle algebras. Finally, Section 4 offers a conclusion and discusses future work.

[3] The formal definition of IVRL is given in Section 2.2. Basically, an IVRL is a residuated lattice on the set of closed intervals of a bounded lattice, in which the subset of intervals consisting of one point is closed under every connective.

2 Algebraic Structures

2.1 Triangle Algebras

To capture the triangular structure of IVRLs, we extend the definition of a residuated lattice with a new constant u ("uncertainty") and two new unary connectives ν ("necessity") and μ ("possibility"); intuitively, the elements of a triangle algebra may be thought of as closed intervals, u as the interval $[0, 1]$, and ν and μ as the operators mapping $[x_1, x_2]$ to $[x_1, x_1]$ and $[x_2, x_2]$ respectively; the formal link with IVRLs will be established in Section 2.2.

Definition 3 A triangle algebra is a structure $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, in which $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice, and in which the following conditions hold:

$T.1$ $\nu x \leq x$,	$T.1'$ $x \leq \mu x$,
$T.2$ $\nu x \leq \nu\nu x$,	$T.2'$ $\mu\mu x \leq \mu x$,
$T.3$ $\nu(x \sqcap y) = \nu x \sqcap \nu y$,	$T.3'$ $\mu(x \sqcap y) = \mu x \sqcap \mu y$,
$T.4$ $\nu(x \sqcup y) = \nu x \sqcup \nu y$,	$T.4'$ $\mu(x \sqcup y) = \mu x \sqcup \mu y$,
$T.5$ $\nu 1 = 1$,	$T.5'$ $\mu 0 = 0$,
$T.6$ $\nu u = 0$,	$T.6'$ $\mu u = 1$,
$T.7$ $\nu\mu x = \mu x$,	$T.7'$ $\mu\nu x = \nu x$,
$T.8$ $\nu(x \Rightarrow y) \leq \nu x \Rightarrow \nu y$,	
$T.9$ $(\nu x \Leftrightarrow \nu y) * (\mu x \Leftrightarrow \mu y) \leq (x \Leftrightarrow y)$,	
$T.10$ $\nu x \Rightarrow \nu y \leq \nu(\nu x \Rightarrow \nu y)$.	

From T.1 and T.2, it is clear that in a triangle algebra, always $\nu\nu x = \nu x$. Similarly, $\mu\mu x = \mu x$. Each of T.3 and T.4 implies that ν is an increasing operator. In the same way, T.3' or T.4' force μ to be increasing too. So, by T.1, T.1', T.2, T.2', T.3, T.4', T.5 and T.5', μ is a closure operator, and ν is an interior operator. Both are also lattice morphisms.

The imposed conditions and used notations can also be found in other algebraic structures. We will compare some of them to each other and to triangle algebras in Section 2.3. Note that T.1'–T.5' are conditions for μ, which is similar to the modal possibility operator; they are dual to T.1–T.5 for ν, which is similar to the modal necessity operator. Only T.4 and T.3' are different: in the modal setting, they are in general not true; and one doesn't want them to be true either (see e.g. [33]). In general, we do not require dependency of μ on ν; a case in which this holds is considered in Proposition 4. The conditions T.6 and T.6' express the complete lack of knowledge about u: its necessity is 0, but its possibility is 1. The conditions T.7 and T.7' are known in modal logics as the S5-principles [26], T.8 as the distribution axiom [18]. Condition T.9 means that an element of a triangle algebra is completely defined by its necessity and possibility: if $\nu x = \nu y$ and $\mu x = \mu y$, then (using Proposition 2(5)) $1 = 1 * 1 = (\nu x \Leftrightarrow \nu y) * (\mu x \Leftrightarrow \mu y) \leq x \Leftrightarrow y$ and therefore $x = y$. An example: using this technique, it is easy to verify that $\nu x \sqcup (u \sqcap \mu x) = x = (u \sqcup \nu x) \sqcap \mu x$. Finally T.10 is a technical condition needed to ensure that triangle algebras correspond to IVRLs. This will be used in Proposition 9 and Theorem 26, in which we will formally prove the equivalence between triangle algebras and IVRLs.

On a residuated lattice, it is possible to define a triangle algebra in which μ depends on ν (and vice versa) if the negation satisfies some properties and a suitable ν can be found:

Proposition 4 *Suppose* $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ *is a residuated lattice such that* $\neg$ *is involutive. If there exists an element* u *in* A *such that* $\neg u = u$*, if* ν *is a unary operator on* A *that satisfies T.1–T.6, T.8, T.10, and if* $(\nu x \Leftrightarrow \nu y) * (\nu \neg x \Leftrightarrow \nu \neg y) \leq (x \Leftrightarrow y)$*, then* $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ *is a triangle algebra if we define* $\mu x = \neg \nu \neg x$.

PROOF.

Some of the properties we have to check, can be proven without using the fact that $\neg$ is involutive:

- From T.1 it follows that $\nu \neg x \leq \neg x$, so $x \leq \neg\neg x \leq \neg \nu \neg x = \mu x$ (using Proposition 2(2,3)). So T.1' holds.
- Proposition 2(2) implies $\nu \neg x \leq \neg\neg \nu \neg x$, so $\nu \neg x \leq \nu \nu \neg x \leq \nu \neg\neg \nu \neg x$ (using T.2 and the monotonicity of ν).
  Therefore $\mu\mu x = \neg \nu \neg\neg \nu \neg x \leq \neg \nu \neg x = \mu x$ (using Proposition 2(3)). So T.2' holds.
- Finally T.5' can be proven by the use of T.5: $\mu 0 = \neg \nu \neg 0 = \neg \nu 1 = \neg 1 = 0$.

Using the condition $\neg u = u$, we can prove by T.6 $\mu u = \neg \nu \neg u = \neg \nu u = \neg 0 = 1$. So T.6' holds. If we apply T.10 on $\neg x$ and 0 (instead of x and y) and T.1, we obtain $\nu \neg \nu \neg x = \neg \nu \neg x$, in other words $\nu \mu x = \mu x$. This is exactly T.7. Because we assumed also that the negation is involutive, we find

- Because $\nu \mu \neg x = \mu \neg x$ by T.7, we know that $\nu \neg \nu x = \neg \nu x$ and therefore

$\nu x = \neg\neg\nu x = \neg\nu\neg\nu x = \mu\nu x$. So T.7' holds.

- Using Proposition 2(9 and 11) and T.4, we find $\mu(x \sqcap y) = \neg\nu\neg(x \sqcap y) = \neg\nu(\neg x \sqcup \neg y) = \neg(\nu\neg x \sqcup \nu\neg y) = \neg\nu\neg x \sqcap \neg\nu\neg y = \mu x \sqcap \mu y$, which is exactly T.3'.
- Analogously also $\mu(x \sqcup y) = \mu x \sqcup \mu y$ holds. This is T.4'.
- Using $x \Rightarrow y = \neg y \Rightarrow \neg x$ (Proposition 2(12)), we find $x \Leftrightarrow y = (x \Rightarrow y) \sqcap (y \Rightarrow x) = (\neg y \Rightarrow \neg x) \sqcap (\neg x \Rightarrow \neg y) = \neg x \Leftrightarrow \neg y$; therefore $(\nu x \Leftrightarrow \nu y) * (\mu x \Leftrightarrow \mu y) = (\nu x \Leftrightarrow \nu y) * (\nu\neg x \Leftrightarrow \nu\neg y) \leq (x \Leftrightarrow y)$. So also T.9 holds.

□

A similar result can be obtained when a suitable μ can be found:

Proposition 5 *Suppose $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice such that $\neg$ is involutive. If there exists an element u in A such that $\neg u = u$, if μ is a unary operator on A that satisfies T.1'–T.6', and if $\mu x * \neg\mu\neg y \leq \mu(x * y)$, $(\mu x \Leftrightarrow \mu y) * (\mu\neg x \Leftrightarrow \mu\neg y) \leq (x \Leftrightarrow y)$ and $\mu(\mu x * \neg\mu y) \leq \mu x * \neg\mu y$, then $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is a triangle algebra if we define $\nu x = \neg\mu\neg x$.*

PROOF. Using $\neg\neg x = x$, $\neg(x * y) = x \Rightarrow \neg y$, $\neg(x \Rightarrow y) = x * \neg y$, $\mu x = \neg\nu\neg x$, $x \leq y$ iff $\neg y \leq \neg x$ and the de Morgan laws (see Proposition 2), it is easy to translate the given properties to T.1–T.6, T.8, T.10 and $(\nu x \Leftrightarrow \nu y) * (\nu\neg x \Leftrightarrow \nu\neg y) \leq (x \Leftrightarrow y)$. So we can apply the previous proposition to verify also T.7, T.7' and T.9. □

In the definition of a triangle algebra, T.8 can be replaced by two other requirements because of the following lemma.

Lemma 6 *Suppose $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice, and ν is a unary operator on L satisfying T.5. Then T.8 is equivalent to the two properties (together) $\nu(x \sqcap y) \leq \nu x$ (A) and $\nu x * \nu y \leq \nu(x * y)$ (B), for all x and y in L.*

PROOF. From T.5, Proposition 2(4) and T.8, we obtain $1 = \nu 1 = \nu((x \sqcap y) \Rightarrow x) \leq \nu(x \sqcap y) \Rightarrow \nu x$, so applying Proposition 2(4) again: $\nu(x \sqcap y) \leq \nu x$ (which is (A)). From Proposition 2(6), we obtain $x = (y \Rightarrow (x * y)) \sqcap x$ and hence $\nu x = \nu((y \Rightarrow (x * y)) \sqcap x)$. Now applying (A) and T.8: $\nu x \leq \nu(y \Rightarrow (x * y)) \leq \nu y \Rightarrow \nu(x * y)$; so $\nu x * \nu y \leq \nu(x * y)$ follows from the residuation principle.
Conversely, let us assume (A) and (B). We will prove that $\nu x * \nu(x \Rightarrow y) \leq \nu y$, which is equivalent to T.8, due to the residuation property. Applying (B), Proposition 2(7) and (A), we obtain successively: $\nu x * \nu(x \Rightarrow y) \leq \nu(x * (x \Rightarrow y)) = \nu(y \sqcap (x * (x \Rightarrow y))) \leq \nu y$. □

Condition (A) in Lemma 6 means exactly that ν is an increasing operator. Therefore it is easy to see that this is a weaker property than T.3, or T.4.

The direct images of a triangle algebra under the necessity and the possibility operators are identical, and every element of this image is invariant under both operators:

Lemma 7 *Let $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra. The following statements are equivalent for all x in A:*

(1) $x = \nu y$ for some y in A,
(2) $x = \mu y$ for some y in A,
(3) $x = \nu x$,
(4) $x = \mu x$,
(5) $\nu x = \mu x$.

PROOF. Let x be in A.

(1) implies (2): If $x = \nu y$, then $x = \mu\nu y$ (using T.7').
(2) implies (3): If $x = \mu z$, then $\nu x = \nu\mu z = \mu z = x$ (using T.7).
(3) implies (4): If $x = \nu x$, then $\mu x = \mu\nu x = \nu x = x$ (using T.7').
(4) implies (5): If $x = \mu x$, then $\nu x = \nu\mu x = \mu x$ (using T.7).
(5) implies (1): Finally, if $\nu x = \mu x$, then $x = \nu x$ because $\nu x \leq x \leq \mu x$ (using T.1 and T.1'). □

Definition 8 *The set of exact elements $E(\mathcal{A})$ of a triangle algebra $\mathcal{A}$ is $\{x \in A \mid \nu x = x\}$.*

As a consequence of Lemma 7 and Definition 3, we find:

Proposition 9 *Let $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra. Then $E(\mathcal{A}) = \nu(A) = \mu(A) = \nu(E(\mathcal{A})) = \mu(E(\mathcal{A}))$. This set contains 0 and 1, but not u (unless in the trivial case when $|A| = 1$).*

The set of exact elements of a triangle algebra is closed under all the defined (unary and binary) operators:

Corollary 10 *Let $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra. Then $E(\mathcal{A})$ is closed under $\sqcap$, $\sqcup$, $*$ and $\Rightarrow$, and therefore $\mathcal{E}(\mathcal{A}) = (E(\mathcal{A}), \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ (in which the binary operators are restricted to $E(\mathcal{A})$) is a residuated lattice. In other words, $\mathcal{E}(\mathcal{A})$ is an algebraic subreduct of $\mathcal{A}$.*

PROOF. By T.3, $\nu(x \sqcap y) = \nu x \sqcap \nu y = x \sqcap y$ for every x and y in $E(\mathcal{A})$. So $E(\mathcal{A})$ is closed under $\sqcap$. Using T.4 we can prove in an analogous way that $E(\mathcal{A})$ is closed under $\sqcup$. The operator ν is a surjective homomorphism from $(A, \sqcap, \sqcup, 0, 1)$ to $(E(\mathcal{A}), \sqcap, \sqcup, 0, 1)$, so the bounded lattice structure is preserved.

From Lemma 6, we know that $\nu x * \nu y \leq \nu(x * y)$. Applying this property and T.1, we obtain for every x and y in $E(\mathcal{A})$ that $\nu(x*y) \leq x*y = \nu x * \nu y \leq \nu(x*y)$ and hence $x * y \in E(\mathcal{A})$.
For x and y in $E(\mathcal{A})$, applying T.1 and T.10, we obtain $x \Rightarrow y = \nu x \Rightarrow \nu y = \nu(\nu x \Rightarrow \nu y) = \nu(x \Rightarrow y)$, which means $x \Rightarrow y \in E(\mathcal{A})$. $\square$

It is possible to express νx and μy in terms of the ordering and $E(\mathcal{A})$:

Proposition 11 *Let* $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ *be a triangle algebra. Then* $\nu x = \sup\{y \in E(\mathcal{A}) \mid y \leq x\}$ *and* $\mu x = \inf\{y \in E(\mathcal{A}) \mid x \leq y\}$.

PROOF. On the one hand, $\nu x \leq \sup\{y \in E(\mathcal{A}) \mid y \leq x\}$ because $\nu x \in \{y \in E(\mathcal{A}) \mid y \leq x\}$. On the other hand, $\sup\{y \in E(\mathcal{A}) \mid y \leq x\} \leq \nu x$ because, for every y in $E(\mathcal{A})$, $y = \nu y \leq \nu x$ if $y \leq x$.
The proof of the second part is analogous. $\square$

It is also possible to describe the ordering on a triangle algebra in terms of the restricted ordering on the set of exact elements:

Proposition 12 *In a triangle algebra* $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, $x \leq y$ *is equivalent with* $\nu x \leq \nu y$ *and* $\mu x \leq \mu y$ *for all* x *and* y *in* A.

PROOF. If $x \leq y$, then $\nu x \leq \nu y$ and $\mu x \leq \mu y$ because ν and μ are increasing. Now suppose $\nu x \leq \nu y$ and $\mu x \leq \mu y$. Then we find, using T.3 and T.3', $\nu(x \sqcap y) = \nu x \sqcap \nu y = \nu x$ and $\mu(x \sqcap y) = \mu x \sqcap \mu y = \mu x$. So by T.9, $x \sqcap y = x$. $\square$

As two special cases, $x \leq u$ iff $\nu x = 0$, and $u \leq x$ iff $\mu x = 1$.

We continue this section with two propositions that examine the relationship between ν and $*$, and μ and $*$.
From Lemma 6, we already know the inequality $\nu x * \nu y \leq \nu(x * y)$. We will now prove that even equality holds.

Proposition 13 *In every triangle algebra* $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, $\nu(x * y) = \nu x * \nu y$ *for every* x *and* y *in* A.

PROOF. Let x and y be in A. Observe that $\nu x \sqcup u = x \sqcup u$, because (using T.4, T.2, T.4' and T.6')

- $\nu(\nu x \sqcup u) = \nu\nu x \sqcup \nu u = \nu x \sqcup \nu u = \nu(x \sqcup u)$, and
- $\mu(\nu x \sqcup u) = \mu\nu x \sqcup \mu u = 1 = \mu x \sqcup \mu u = \mu(x \sqcup u)$.

Hence, by T.9, $\nu x \sqcup u$ and $x \sqcup u$ must be equal.
Furthermore $\nu(x * u) = 0$, because $x * u \leq u$ and $\nu u = 0$.
Therefore, by T.4 and Proposition 2(8), $\nu(x * y) = \nu(x * y) \sqcup \nu(x * u) = \nu((x * y) \sqcup (x * u)) = \nu(x * (y \sqcup u))$. By a symmetric argument we find

$$\nu(x * y) = \nu((x \sqcup u) * (y \sqcup u)).$$

Using this equality (twice) and $x \sqcup u = \nu x \sqcup u$, we can now conclude that $\nu(x * y) = \nu((x \sqcup u) * (y \sqcup u)) = \nu((\nu x \sqcup u) * (\nu y \sqcup u)) = \nu(\nu x * \nu y) \leq \nu x * \nu y$. The other inequality ($\nu x * \nu y \leq \nu(x * y)$) follows from Lemma 6. □

A similar equality for μ is in general not true, although the inequality $\mu(x*y) \leq \mu x * \mu y$ always holds (see the proof of Proposition 14). But we do have an equivalent formulation for this property:

Proposition 14 *In every triangle algebra* $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, *the following properties are equivalent:*

(1) $\mu(x * y) = \mu x * \mu y$ *for every* x *and* y *in* A, *and*
(2) $\mu(x * y) = \mu(x * z)$ *if* $\mu y = \mu z$, *for every* x, y *and* z *in* A.

PROOF. (1) implies (2): if $\mu y = \mu z$, $\mu(x * y) = \mu x * \mu y = \mu x * \mu z = \mu(x * z)$.
(2) implies (1): by T.1' and T.2' $\mu\mu y = \mu y$. Using (2), we find $\mu(x*\mu y) = \mu(x*y)$. Similarly, $\mu(x*\mu y) = \mu(\mu x*\mu y)$. Therefore $\mu x*\mu y \leq \mu(\mu x*\mu y) = \mu(x*y)$.
The inequality $\mu(x * y) \leq \mu x * \mu y$ always holds, because, using Proposition 9, $\mu x * \mu y$ is in $E(\mathcal{A})$ and therefore $\mu(x * y) \leq \mu(\mu x * \mu y) = \mu x * \mu y$. □

We conclude this section with a proposition about the value of $\neg u$:

Proposition 15 *Suppose* $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ *is a triangle algebra. Then* $\neg u \leq u$. *If* $\neg$ *is involutive, then* $\neg u = u$ *and* $u * u = 0$.

PROOF. Because of T.1, $\nu\neg u \leq \neg u$. Using Proposition 2(2 and 3), we find $u \leq \neg\neg u \leq \neg\nu\neg u$. But $\nu\neg u$ is in $E(\mathcal{A})$, therefore also $\neg\nu\neg u$ is in $E(\mathcal{A})$ (by definition of $\neg$ and Proposition 9). So $1 = \mu u \leq \mu\neg\nu\neg u = \neg\nu\neg u$, which implies $\nu\neg u \leq \neg\neg\nu\neg u \leq \neg 1 = 0$. This is equivalent to $\neg u \leq u$.
If $\neg$ is involutive, we can prove in an analogous way (starting from $\neg u \leq \mu\neg u$) that $u \leq \neg u$, which is equivalent to $u * u = 0$ because of the residuation principle. □

2.2 Connection to Interval-Valued Residuated Lattices

Definition 16 Given a lattice $\mathcal{L} = (L, \sqcap, \sqcup)$, its triangularization $\mathbb{T}(\mathcal{L})$ is the structure $\mathbb{T}(\mathcal{L}) = (Int(\mathcal{L}), \sqcap, \sqcup)$ defined by

- $Int(\mathcal{L}) = \{[x_1, x_2] \mid (x_1, x_2) \in L^2 \text{ and } x_1 \leq x_2\}$
- $[x_1, x_2] \bigsqcap [y_1, y_2] = [x_1 \sqcap y_1, x_2 \sqcap y_2]$
- $[x_1, x_2] \bigsqcup [y_1, y_2] = [x_1 \sqcup y_1, x_2 \sqcup y_2]$

The set $D_{\mathcal{L}} = \{[x, x] \mid x \in L\}$ is called the diagonal of $\mathbb{T}(\mathcal{L})$.
The first and the second projection pr_1 and pr_2 are the mappings from $T(\mathcal{L})$ to L, defined by $pr_1([x_1, x_2]) = x_1$ and $pr_2([x_1, x_2]) = x_2$, for all $[x_1, x_2]$ in $T(\mathcal{L})$.

As an example, $\mathcal{L}^I$ is the triangularization of $([0, 1], \min, \max)$. Its diagonal is shown in Figure 1 as the hypotenuse of the triangle.

Proposition 17 *If $\mathcal{L}$ is a lattice, then $\mathbb{T}(\mathcal{L})$ is again a lattice. If $\mathcal{L}$ contains a smallest element 0 (resp. a greatest element 1), then $\mathbb{T}(\mathcal{L})$ has $[0, 0]$ as smallest element (resp. $[1, 1]$ as greatest element). Moreover, if $\mathcal{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice, then it is always possible to construct a residuated lattice on $\mathbb{T}(\mathcal{L})$.*

PROOF. Because $\bigsqcap$ and $\bigsqcup$ are defined as componentwise extensions of $\sqcap$ and $\sqcup$, the lattice properties are inherited. If $0 \leq x$ for every x in L, then $[0, 0] \leq [x_1, x_2]$ for every $[x_1, x_2]$ in $Int(\mathcal{L})$ (and similarly for 1).
Now suppose $\mathcal{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice and define

$$[x_1, x_2] \odot [y_1, y_2] = [x_1 * y_1, x_2 * y_2] \quad (1)$$

$$[x_1, x_2] \Rightarrow_{\odot} [y_1, y_2] = [(x_1 \Rightarrow y_1) \sqcap (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2.] \quad (2)$$

We prove that the structure $(Int(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_{\odot}, [0, 0], [1, 1])$ is a residuated lattice too:

- commutativity and associativity of $\odot$ follow immediately from commutativity and associativity of $*$, because $\odot$ is the componentwise extension of $*$,
- the greatest element $[1,1]$ is a neutral element for $\odot$: $[x_1,x_2]\odot[1,1]=[x_1*1,x_2*1]=[x_1,x_2]$,
- the residuation principle: $[x_1,x_2]\odot[y_1,y_2]\leq[z_1,z_2]$ iff $[x_1*y_1,x_2*y_2]\leq[z_1,z_2]$ iff ($x_1*y_1\leq z_1$ and $x_2*y_2\leq z_2$) iff ($x_1\leq y_1\Rightarrow z_1$ and $x_2\leq y_2\Rightarrow z_2$) iff (because $x_1\leq x_2$ and possibly $y_1\Rightarrow z_1\not\leq y_2\Rightarrow z_2$) $[x_1,x_2]\leq[(y_1\Rightarrow z_1)\sqcap(y_2\Rightarrow z_2),y_2\Rightarrow z_2]$ iff $[x_1,x_2]\leq[y_1,y_2]\Rightarrow_\odot[z_1,z_2]$.

□

Formulas (1) and (2) are not the only possible way of defining residuated lattices on $Int(\mathcal{L})$; Example 18 gives a case in point on $\mathcal{L}^I$.

Example 18 *As mentioned in Section 1.1, a t-norm on* $[0,1]$ *is residuated iff it is left-continuous; this property however does not extend to* $\mathcal{L}^I$ *[9]. While a general characterization of residuated t-norms on* $\mathcal{L}^I$ *has not yet been found, it was shown in [8] that if* T *induces a residuated lattice on* $[0,1]$*, then for each* α *in* $[0,1]$*, the mapping* $\mathcal{T}_{T,\alpha}$ *defined by, for* $x=[x_1,x_2]$ *and* $y=[y_1,y_2]$ *in* L^I*,*

$$\mathcal{T}_{T,\alpha}(x,y)=[T(x_1,y_1),\max(T(\alpha,T(x_2,y_2)),T(x_1,y_2),T(x_2,y_1))], \tag{3}$$

induces a residuated lattice on $\mathcal{L}^I$*, with residual implicator*

$$\mathcal{I}_{\mathcal{T}_{T,\alpha}}(x,y)=[\min(I_T(x_1,y_1),I_T(x_2,y_2)),\min(I_T(T(x_2,\alpha),y_2),I_T(x_1,y_2))].$$

If we define $\nu x=[x_1,x_1]$ *and* $\mu x=[x_2,x_2]$ *for all* $x=[x_1,x_2]$ *in* L^I*, then* $(L^I,\sqcap,\sqcup,\mathcal{T}_{T,\alpha},\mathcal{I}_{\mathcal{T}_{T,\alpha}},\nu,\mu,[0,0],[0,1],[1,1])$ *is a triangle algebra, as we will later prove (Example 28). Moreover, from Propositions 19 and 21 it will follow that this is the only way to construct a triangle algebra on a residuated lattice on* $\mathcal{L}^I$*.*

Two important values of α *can be distinguished in Formula (3):*

- *If* $\alpha=1$*, we obtain t-representable t-norms on* $\mathcal{L}^I$*:*
  $\mathcal{T}_{T,1}(x,y)=[T(x_1,y_1),T(x_2,y_2)]$*, which can be seen as the straightforward (and most commonly used) extension of* T *to* $\mathcal{L}^I$*. Because of Proposition 14, these residuated t-norms on* $\mathcal{L}^I$ *are characterized by the property* $\mu(\mathcal{T}(x,y))=\mathcal{T}(\mu x,\mu y)$*, if* μ *is defined in the usual way (*$\mu[x_1,x_2]=[x_2,x_2]$*).*
- *If* $\alpha=0$*, we obtain pseudo t-representable t-norms on* $\mathcal{L}^I$*:*
  $\mathcal{T}_{T,0}(x,y)=[T(x_1,y_1),\max(T(x_1,y_2),T(x_2,y_1))]$*. These t-norms are inherently more complex than their t-representable counterparts, but, as we shall see at the end of this section, satisfy more relevant properties.*

Given a residuated lattice on a triangularization, we will show that we can always extend it to a triangle algebra. However, when we choose u to be $[0,1]$, there is only one possible way to do this, due to the following proposition.

Proposition 19 *If* $(Int(\mathcal{L}), \sqcap, \sqcup, \odot, \Rightarrow_\odot, \nu, \mu, [0,0], [0,1], [1,1])$ *is a triangle algebra on a triangularization* $(Int(\mathcal{L}), \sqcap, \sqcup)$ *of a bounded lattice* $\mathcal{L}$*, then* $\nu[x_1,x_2] = [x_1,x_1]$ *and* $\mu[x_1,x_2] = [x_2,x_2]$ *for every* $[x_1,x_2]$ *in* $Int(\mathcal{L})$*.*

PROOF. We first show that $\nu[x_1,x_2] = \nu[x_1,1]$ for every $[x_1,x_2]$ in $Int(\mathcal{L})$, using T.6 and T.4: $\nu[x_1,x_2] = \nu[x_1,x_2] \sqcup [0,0] = \nu[x_1,x_2] \sqcup \nu[0,1] = \nu([x_1,x_2] \sqcup [0,1]) = \nu[x_1,1]$. So $\nu[x_1,x_2]$ does not depend on x_2.
Suppose there exists an interval $[a_1,a_2]$ in $Int(\mathcal{L})$ for which $\nu[a_1,a_2] = [b_1,b_2]$ and $[b_1,b_2] \neq [a_1,a_1]$. Because $[b_1,b_2] = \nu[a_1,a_2] = \nu[a_1,a_1] \leq [a_1,a_1]$, we have $b_1 < a_1$ (if b_1 were equal to a_1, then $a_1 = b_1 \leq b_2 \leq a_1$, which would imply $[b_1,b_2] = [a_1,a_1]$).
We can now show that $\nu[a_1,1] = \nu[b_1,1]$ and $\mu[a_1,1] = \mu[b_1,1]$, which is in contradiction with T.9 because of Proposition 2(5) and $[a_1,1] \neq [b_1,1]$.

- $\nu[a_1,1] = \nu[a_1,a_2] = \nu\nu[a_1,a_2] = \nu[b_1,b_2] = \nu[b_1,1]$ (using T.2)
- Because $[0,1] \leq [b_1,1] < [a_1,1]$, $[1,1] = \mu[0,1] \leq \mu[b_1,1] \leq \mu[a_1,1]$.

So we conclude that our assumption was false. There exists no interval $[a_1,a_2]$ in $Int(\mathcal{L})$ such that $\nu[a_1,a_2] \neq [a_1,a_1]$.
The proof for μ is completely analogous. □

Remark that in this case ($\nu[x_1,x_2] = [x_1,x_1]$), the set of exact elements $E((Int(\mathcal{L}), \sqcap, \sqcup, \odot, \Rightarrow_\odot, \nu, \mu, [0,0], [0,1], [1,1]))$ coincides with $D_\mathcal{L}$.
It is not always true that $u = [0,1]$ if $(Int(\mathcal{L}), \sqcap, \sqcup, \odot, \Rightarrow_\odot, \nu, \mu, [0,0], u, [1,1])$ is a triangle algebra on a triangularization $(Int(\mathcal{L}), \sqcap, \sqcup)$ of a bounded lattice $\mathcal{L}$. A counterexample can be given on the triangularization of a bounded (linear) lattice $\mathcal{L}$ with three elements, say 0, 1/2 and 1. If we define ν and μ on this triangularization by $\nu[0,0] = \nu[0,1/2] = \nu[1/2,1/2] = \mu[0,0] = [0,0]$, $\nu[0,1] = \nu[1/2,1] = \mu[0,1] = \mu[0,1/2] = [0,1]$ and $\nu[1,1] = \mu[1/2,1/2] = \mu[1/2,1] = \mu[1,1] = [1,1]$, then $(Int(\mathcal{L}), \sqcap, \sqcup, \sqcap, \Rightarrow_\sqcap, \nu, \mu, [0,0], [1/2,1/2], [1,1])$ is a triangle algebra in which $u \neq [0,1]$.
We will prove, however, that in many cases there are no counterexamples.

It is easy to see that 0 is $\sqcap$-irreducible[4] in $\mathcal{L}$ if and only if $[0,0]$ is $\sqcap$-irreducible in $\mathbb{T}(\mathcal{L})$, and similarly for the $\sqcup$-irreducibility of 1. We use this in the following lemma.

[4] This means that $x \sqcap y = 0$ if and only if $x = 0$ or $y = 0$, for every x and y in L. Analogously, 1 is $\sqcup$-irreducible when $x \sqcup y = 1$ iff $x = 1$ or $y = 1$.

Lemma 20 *If* $(Int(\mathcal{L}), \sqcap, \sqcup, \odot, \Rightarrow_\odot, \nu, \mu, [0,0], u, [1,1])$ *is a triangle algebra on a triangularization* $(Int(\mathcal{L}), \sqcap, \sqcup)$ *of a bounded lattice* $\mathcal{L} = (L, \sqcap, \sqcup, 0, 1)$ *in which* 0 *is* $\sqcap$*-irreducible and* 1 $\sqcup$*-irreducible, then* $u = [0,1]$ *or* $u = [a,a]$ *for some* a *in* L*.*

PROOF. Suppose that $u = [u_1, u_2]$, with $u_1 < u_2$. Then $u \neq [u_2, u_2]$. So, according to T.9, it cannot be both the case that $\nu[u_2, u_2] = [0,0]$ and $\mu[u_2, u_2] = [1,1]$ (because of T.6 and T.6'). As we know $[1,1] = \mu u \leq \mu[u_1, u_2]$ (because μ is increasing), necessarily $\nu[u_2, u_2] > [0,0]$. Similarly $\mu[u_1, u_1] < [1,1]$.
From $[0,0] = \nu[0, u_2] = \nu([0,1] \sqcap [u_2, u_2]) = \nu[0,1] \sqcap \nu[u_2, u_2]$ and the fact that $[0,0]$ is $\sqcap$-irreducible, we can then derive that $\nu[0,1] = [0,0]$; and from $[1,1] = \mu[u_1, 1] = \mu([0,1] \sqcup [u_1, u_1]) = \mu[0,1] \sqcup \mu[u_1, u_1]$ and the fact that $[1,1]$ is $\sqcup$-irreducible, that $\mu[0,1] = [1,1]$.
This means that $\nu[0,1] = \nu u$ and $\mu[0,1] = \mu u$, so by T.9 we obtain $[0,1] = u$.
□

Proposition 21 *If* $(Int(\mathcal{L}), \sqcap, \sqcup, \odot, \Rightarrow_\odot, \nu, \mu, [0,0], u, [1,1])$ *is a triangle algebra on a triangularization* $(Int(\mathcal{L}), \sqcap, \sqcup)$ *of a bounded linear lattice* $\mathcal{L}$ *with at least four elements, then* $u = [0,1]$*.*

PROOF. Because in every linear bounded lattice 0 is $\sqcap$-irreducible and 1 $\sqcup$-irreducible, we know from Lemma 20 that if u is not $[0,1]$, then it must be of the form $[a,a]$, with $0 < a < 1$. We will prove that this is impossible. So assume $u = [a,a]$. There exists at least one element in L different from 0, a and 1, say b. We will suppose $0 < a < b < 1$ (the case $0 < b < a < 1$ is analogous). Then $[1,1] = \mu[a,a] \leq \mu[a,b]$. So $\mu[a,b] = \mu[a,a]$. For $\nu[a,b]$ (as for any element of $Int(\mathcal{L})$), there are two possibilities: $[0,a] \leq \nu[a,b]$ or $\nu[a,b] < [a,a]$. The first one can not be the case, because then $[0,1] \leq \mu[0,1] = \mu[0,1] \sqcap \mu[a,a] = \mu[0,a] \leq \mu\nu[a,b] = \nu[a,b] \leq [a,b]$ (a contradiction, as $b < 1$). Therefore $\nu[a,b]$ has to be strictly smaller than $[a,a]$. In this case, $\nu[a,b] = \nu\nu[a,b] \leq \nu[a,a] = [0,0]$. But then $\nu[a,b] = \nu[a,a]$. As we already know $\mu[a,b] = \mu[a,a]$ too, T.9 would imply that $[a,b] = [a,a]$, a contradiction. So our assumption that u was not $[0,1]$, must be false. □

Definition 22 An interval-valued residuated lattice (IVRL) is a residuated lattice $(Int(\mathcal{L}), \sqcap, \sqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ on the triangularization $\mathbb{T}(\mathcal{L})$ of a bounded lattice $\mathcal{L}$, in which the diagonal $D_\mathcal{L}$ is closed under $\odot$ and $\Rightarrow_\odot$, i.e., $[x_1, x_1] \odot [y_1, y_1] \in D_\mathcal{L}$ and $[x_1, x_1] \Rightarrow_\odot [y_1, y_1] \in D_\mathcal{L}$ for x_1, y_1 in L.
When we add $[0,1]$ as a constant, and p_v and p_h (defined by $p_v([x_1, x_2]) = [x_1, x_1]$ and $p_h([x_1, x_2]) = [x_2, x_2]$, for all $[x_1, x_2]$ in $Int(\mathcal{L})$) as unary operators, the structure $(Int(\mathcal{L}), \sqcap, \sqcup, \odot, \Rightarrow_\odot, p_v, p_h, [0,0], [0,1], [1,1])$ is called an extended IVRL.

Remark 23 Note that in IVRLs $pr_1([x_1,x_1]\odot[y_1,y_1]) = pr_2([x_1,x_1]\odot[y_1,y_1])$ and $pr_1([x_1,x_1]\Rightarrow_\odot [y_1,y_1]) = pr_2([x_1,x_1]\Rightarrow_\odot [y_1,y_1])$. Therefore $(D_\mathcal{L},\sqcap,\sqcup,\odot,\Rightarrow_\odot,[0,0],[1,1])$ (in which the binary operators are restricted to $D_\mathcal{L}$) is isomorphic to $(L,\sqcap,\sqcup,*,\Rightarrow,0,1)$, if we define $x*y = pr_1([x,x]\odot[y,y])$ and $x\Rightarrow y = pr_1([x,x]\Rightarrow_\odot [y,y])$ for all x and y in L.
In general, this is not true for all residuated lattices on a triangularization, as the next example shows.

Example 24

- *With the t-norm $\mathcal{T}_{T_P,T_M}$, which is defined by $\mathcal{T}_{T_P,T_M}([x_1,x_2],[y_1,y_2]) = [x_1y_1, \min(x_2,y_2)]$, and its residual implicator $\mathcal{I}_{T_P,T_M}$, the structure $(L^I,\sqcap,\sqcup,\mathcal{T}_{T_P,T_M}, \mathcal{I}_{T_P,T_M},[0,0],[1,1])$ is a residuated lattice on $\mathcal{L}^I$. But $\mathcal{T}_{T_P,T_M}([0.5,0.5],[0.3,0.3]) = [0.15,0.3]$, so according to our definition, it is not an IVRL because $D_\mathcal{L}$ is not closed under $\mathcal{T}_{T_P,T_M}$ (although it is closed under $\mathcal{I}_{T_P,T_M}$).*
- *For $\mathcal{T}_1$, defined[5] by $\mathcal{T}_1([x_1,x_2],[y_1,y_2]) = [\max(0,x_1+y_1-1),\max(0,2x_1+y_2-2,2y_1+x_2-2,x_1+y_1-1)]$, it is the other way around: $D_\mathcal{L}$ is closed under this t-norm, but not under its residual implicator $\mathcal{I}_1$, which is given by $\mathcal{I}_1([x_1,x_2],[y_1,y_2]) = [\min(1,1+y_1-x_1,1+(y_2-x_2)/2),\min(1,2+y_2-2x_1)]$. Also in this case however, $(L^I,\sqcap,\sqcup,\mathcal{T}_1,\mathcal{I}_1,[0,0],[1,1])$ is a residuated lattice.*

The next important theorem establishes triangle algebras as the equational representation of IVRLs.
In the proof of this theorem, we will use the following lemma, adapted from [8]:

Lemma 25 *In an IVRL $(Int(\mathcal{L}),\sqcap,\sqcup,\odot,\Rightarrow_\odot,[0,0],[1,1])$, the first component of $[x_1,x_2]\odot[y_1,y_2]$ is independent of x_2 and y_2, for every $[x_1,x_2]$ and $[y_1,y_2]$ in $Int(\mathcal{L})$.*

Theorem 26 *There is a one-to-one correspondence between the class of IVRLs and the class of triangle algebras. Every extended IVRL is a triangle algebra and conversely, every triangle algebra is isomorphic to an extended IVRL.*

PROOF. We will use the notations X for the class of triangle algebras, and Y for the class of IVRLs.
We will define mappings ϕ: $X \longrightarrow Y$ and ψ: $Y \longrightarrow X$ and show

(1) that $\phi(x)$ is indeed in Y for any x in X,
(2) that $\psi(y)$ is indeed in X for any y in Y,
(3) that $\psi(\phi(x))$ is isomorphic to x for any x in X, and
(4) that $\phi(\psi(y))$ is isomorphic to y for any y in Y.

[5] This example is from [9] (Example 8.1), translated from intuitionistic fuzzy set theory to interval-valued fuzzy set theory.

Because the image $\psi(y)$ of y will be the extended IVRL of y, this will prove the desired equivalence.
We define $\phi(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1) = (Int(\mathcal{E}(\mathcal{A})), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ in the following way: $Int(\mathcal{E}(\mathcal{A}))$, $\bigsqcap$ and $\bigsqcup$ are defined as in Definition 16, $\odot$ is a binary operator defined as $[x_1, x_2] \odot [y_1, y_2] = [\nu((x_1 \sqcup (u \sqcap x_2)) * (y_1 \sqcup (u \sqcap y_2))), \mu((x_1 \sqcup (u \sqcap x_2)) * (y_1 \sqcup (u \sqcap y_2)))]$, for all $[x_1, x_2]$ and $[y_1, y_2]$ in $Int(\mathcal{E}(\mathcal{A}))$, and $\Rightarrow_\odot$ is a binary operator defined as

$[x_1, x_2] \Rightarrow_\odot [y_1, y_2] = [\nu((x_1 \sqcup (u \sqcap x_2)) \Rightarrow (y_1 \sqcup (u \sqcap y_2))), \mu((x_1 \sqcup (u \sqcap x_2)) \Rightarrow (y_1 \sqcup (u \sqcap y_2)))]$

for all $[x_1, x_2]$ and $[y_1, y_2]$ in $Int(\mathcal{E}(\mathcal{A}))$ (in which the ordering is defined as usual: $[x_1, x_2] \leq [y_1, y_2]$ iff $[x_1, x_2] \bigsqcap [y_1, y_2] = [x_1, x_2]$).
And we also define $\psi(A, \sqcap, \sqcup, *, \Rightarrow, [0,0], [1,1]) = (A, \sqcap, \sqcup, *, \Rightarrow, p_v, p_h, [0,0], [0,1], [1,1])$, in which p_v and p_h are defined by $p_v([x_1, x_2]) = [x_1, x_1]$ and $p_h([x_1, x_2]) = [x_2, x_2]$, for all $[x_1, x_2]$ in A. In other words, the image is the extended IVRL.
Furthermore, if $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is a triangle algebra, we define the mapping χ: $A \longrightarrow Int(\mathcal{E}(\mathcal{A}))$ as $\chi(x) = [\nu x, \mu x]$. Remark that $\chi(0) = [0,0]$, $\chi(u) = [0,1]$ and $\chi(1) = [1,1]$ (using T.1, T.1', T.5, T.5', T.6 and T.6'). Because $[x_1, x_2] = \chi(x_1 \sqcup (u \sqcap x_2))$ for any $[x_1, x_2]$ in $Int(\mathcal{E}(\mathcal{A}))$, χ is a surjection. And it is an injection too, because of T.9.

(1) We verify that $\phi(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is an IVRL if $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is a triangle algebra. Notice that, because of conditions T.3, T.3', T.4 and T.4', $\chi(x \sqcap y) = [\nu(x \sqcap y), \mu(x \sqcap y)] = [\nu x \sqcap \nu y, \mu x \sqcap \mu y] = [\nu x, \mu x] \bigsqcap [\nu y, \mu y] = \chi(x) \bigsqcap \chi(y)$ and analogously $\chi(x \sqcup y) = \chi(x) \bigsqcup \chi(y)$ for all x and y in A. Also notice that $\chi(x * y) = \chi((\nu x \sqcup (u \sqcap \mu x)) * (\nu y \sqcup (u \sqcap \mu y))) = [\nu x, \mu x] \odot [\nu y, \mu y] = \chi(x) \odot \chi(y)$ for any x and y in A, and, completely analogous, that $\chi(x \Rightarrow y) = \chi(x) \Rightarrow_\odot \chi(y)$.
From these properties we conclude that $(Int(\mathcal{E}(\mathcal{A})), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ is isomorphic to $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$, and therefore a residuated lattice. We only need to verify that the diagonal of $Int(\mathcal{E}(\mathcal{A}))$ is closed under $\odot$ and $\Rightarrow_\odot$. Indeed, $[x, x] \odot [y, y] = [\nu((x \sqcup (u \sqcap x)) * (y \sqcup (u \sqcap y))), \mu((x \sqcup (u \sqcap x)) * (y \sqcup (u \sqcap y)))] = [\nu(x * y), \mu(x * y)] = [x * y, x * y]$ because x and y belong to of $E(\mathcal{A})$. The verification for $\Rightarrow_\odot$ is completely analogous.

(2) Suppose $\mathbb{T}(\mathcal{L}) = (Int(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ is an IVRL and that p_v and p_h are defined by $p_v[x_1, x_2] = [x_1, x_1]$ and $p_h[x_1, x_2] = [x_2, x_2]$. We prove that $\psi(\mathbb{T}(\mathcal{L})) = (Int(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, p_v, p_h, [0,0], [0,1], [1,1])$ is a triangle algebra.
We will denote the restrictions of $\odot$ and $\Rightarrow_\odot$ to $D_{\mathbb{T}(\mathcal{L})}$ as $*$ and $\Rightarrow$. In more detail: $[x, x] \odot [y, y] = [x * y, x * y]$ and $[x, x] \Rightarrow_\odot [y, y] = [x \Rightarrow y, x \Rightarrow y]$.
   - The conditions T.1–T.7 and T.1'–T.7' are verified easily.
   - Because T.3 and T.5 are valid, for the proof of T.8 it suffices to show (see Lemma 6) that $p_v(x) \odot p_v(y) \leq p_v(x \odot y)$ for every x and y in $Int(\mathcal{L})$.

This is true because $p_v(x) \odot p_v(y) = [x_1 * y_1, x_1 * y_1] = p_v[x_1 * y_1, x_1 * y_1] = p_v(p_v(x) \odot p_v(y)) \le p_v(x \odot y)$.

- For T.9, we will prove that $((p_v(x) \Leftrightarrow_\odot p_v(y)) \odot (p_h(x) \Leftrightarrow_\odot p_h(y))) \odot x \le y$, which is equivalent with $(p_v(x) \Leftrightarrow_\odot p_v(y)) \odot (p_h(x) \Leftrightarrow_\odot p_h(y)) \le x \Rightarrow_\odot y$ by the residuation principle.

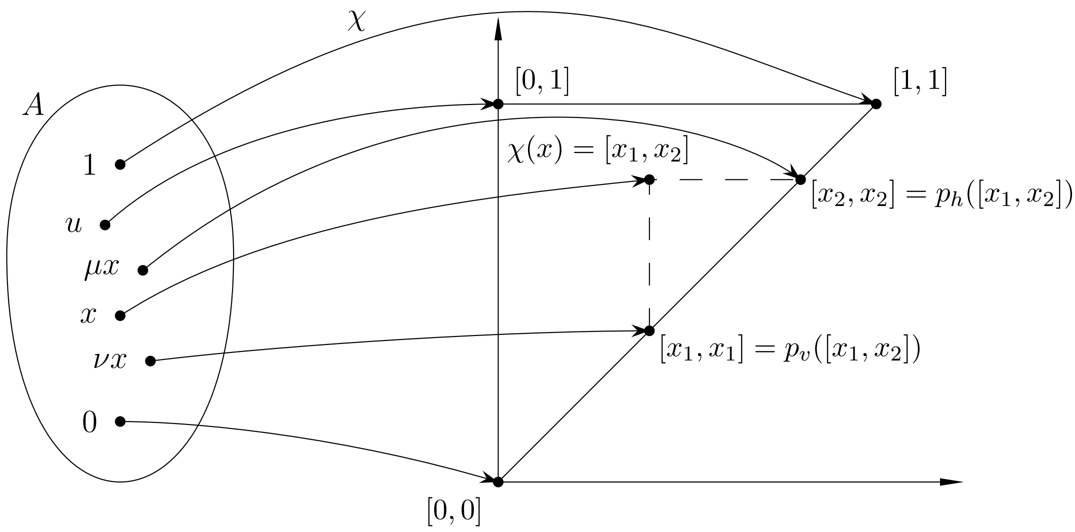
$$
\begin{aligned}
&((p_v(x) \Leftrightarrow_\odot p_v(y)) \odot (p_h(x) \Leftrightarrow_\odot p_h(y))) \odot x \\
=&((([x_1, x_1] \Rightarrow_\odot [y_1, y_1]) \textstyle\bigsqcap([y_1, y_1] \Rightarrow_\odot [x_1, x_1])) \odot \\
&\quad (([x_2, x_2] \Rightarrow_\odot [y_2, y_2]) \textstyle\bigsqcap([y_2, y_2] \Rightarrow_\odot [x_2, x_2]))) \odot [x_1, x_2] \\
=&(([x_1 \Rightarrow y_1, x_1 \Rightarrow y_1] \textstyle\bigsqcap[y_1 \Rightarrow x_1, y_1 \Rightarrow x_1]) \odot \\
&\quad ([x_2 \Rightarrow y_2, x_2 \Rightarrow y_2] \textstyle\bigsqcap[y_2 \Rightarrow x_2, y_2 \Rightarrow x_2])) \odot [x_1, x_2] \\
=&([x_1 \Leftrightarrow y_1, x_1 \Leftrightarrow y_1] \odot [x_2 \Leftrightarrow y_2, x_2 \Leftrightarrow y_2]) \odot [x_1, x_2] \\
=&[(x_1 \Leftrightarrow y_1) * (x_2 \Leftrightarrow y_2), (x_1 \Leftrightarrow y_1) * (x_2 \Leftrightarrow y_2)] \odot [x_1, x_2].
\end{aligned}
$$

  The second component of $((p_v(x) \Leftrightarrow_\odot p_v(y)) \odot (p_h(x) \Leftrightarrow_\odot p_h(y))) \odot x$ is therefore smaller than or equal to the second component of $[(x_1 \Leftrightarrow y_1) * (x_2 \Leftrightarrow y_2), (x_1 \Leftrightarrow y_1) * (x_2 \Leftrightarrow y_2)] \odot [x_2, x_2]$, which is equal to $((x_1 \Leftrightarrow y_1) * (x_2 \Leftrightarrow y_2)) * x_2$. This expression is smaller than or equal to y_2: $((x_1 \Leftrightarrow y_1) * (x_2 \Leftrightarrow y_2)) * x_2 \le (x_2 \Leftrightarrow y_2) * x_2 \le (x_2 \Rightarrow y_2) * x_2 \le y_2$ (Proposition 2(7)).
  The first component of $((p_v(x) \Leftrightarrow_\odot p_v(y)) \odot (p_h(x) \Leftrightarrow_\odot p_h(y))) \odot x$ is, thanks to Lemma 25, equal to the first component of $[(x_1 \Leftrightarrow y_1) * (x_2 \Leftrightarrow y_2), (x_1 \Leftrightarrow y_1) * (x_2 \Leftrightarrow y_2)] \odot [x_1, x_1]$, which is $((x_1 \Leftrightarrow y_1) * (x_2 \Leftrightarrow y_2)) * x_1$. This is smaller than or equal to y_1, so we can conclude that indeed $((p_v(x) \Leftrightarrow p_v(y)) \odot (p_h(x) \Leftrightarrow p_h(y))) \odot x \le y$.
  Analogously also $(p_v(x) \Leftrightarrow_\odot p_v(y)) \odot (p_h(x) \Leftrightarrow_\odot p_h(y)) \le y \Rightarrow_\odot x$ holds, so $(p_v(x) \Leftrightarrow_\odot p_v(y)) \odot (p_h(x) \Leftrightarrow_\odot p_h(y)) \le x \Leftrightarrow_\odot y$.
- Finally we check T.10: $p_v(x) \Rightarrow_\odot p_v(y) = [x_1 \Rightarrow y_1, x_1 \Rightarrow y_1] = p_v([x_1 \Rightarrow y_1, x_1 \Rightarrow y_1]) = p_v(p_v(x) \Rightarrow_\odot p_v(y))$.

(3) Now we show that $\psi(\phi(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1))$ is isomorphic to $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$. We use again the mapping χ. Looking at part 1 of this proof, we see that the only thing we still need to prove, is that $\chi(\nu x) = p_v(\chi(x))$ and $\chi(\mu x) = p_h(\chi(x))$ for all x in A. Indeed: $\chi(\nu x) = [\nu\nu x, \mu\nu x] = [\nu x, \nu x] = p_v[\nu x, \mu x] = p_v(\chi(x))$ and analogously $\chi(\mu x) = p_h(\chi(x))$.

(4) Finally we verify that $\phi(\psi(A, \sqcap, \sqcup, *, \Rightarrow, [0, 0], [1, 1]))$ is isomorphic to the IVRL $(A, \sqcap, \sqcup, *, \Rightarrow, [0, 0], [1, 1])$. From part 2 of this proof, we know that $\psi(A, \sqcap, \sqcup, *, \Rightarrow, [0, 0], [1, 1])$ is a triangle algebra. In this triangle algebra, the exact elements are the intervals of the form $[x, x]$. In this case, the mapping χ is defined as $\chi[x, y] = [p_v[x, y], p_h[x, y]] = [[x, x], [y, y]]$. It is indeed an isomorphism: this is exactly what was proven (for general triangle algebras) in part 1 of this proof. □

The isomorphism χ of Theorem 26 is depicted in Figure 2.



Triangle algebra
$(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$

Isomorphic triangle algebra
$(A', \sqcap', \sqcup', *', \Rightarrow', \nu', \mu', [0, 0], [0, 1], [1, 1])$
in which $(A', \sqcap', \sqcup', *', \Rightarrow', [0, 0], [1, 1])$
is an IVRL

Fig. 2. The isomorphism χ from a triangle algebra to an IVRL.

Remark 27 *The equivalence in Theorem 26 allows to establish a correspondence between Lemma 25 and Proposition 13. Indeed, if we take an IVRL* $(Int(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0, 0], [1, 1])$ *as in Lemma 25 and define* ν *and* μ *as in Theorem 26, the second part of this theorem ensures that* $(Int(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, \nu, \mu, [0, 0], [0, 1], [1, 1])$ *is a triangle algebra. We can therefore apply Proposition 13:* $\nu([x_1, x_2] \odot [y_1, y_2]) = \nu[x_1, x_2] \odot \nu[y_1, y_2] = [x_1, x_1] \odot [y_1, y_1]$*, which means exactly that the first component of* $[x_1, x_2] \odot [y_1, y_2]$ *is independent of* x_2 *and* y_2*. Conversely, consider a triangle algebra* $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$*. Now take the isomorphic triangle algebra* $(T(\mathcal{E}(\mathcal{A})), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, p_v, p_h, [0, 0], [0, 1], [1, 1])$ *of the third part of Theorem 26. Because* $(T(\mathcal{E}(\mathcal{A})), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0, 0], [1, 1])$ *is an IVRL, we can apply Lemma 25 and find* $p_v([x_1, x_2] \odot [y_1, y_2]) = p_v(p_v[x_1, x_2] \odot p_v[y_1, y_2])$ *(for every* $[x_1, x_2]$ *and* $[y_1, y_2]$ *in* $T(\mathcal{E}(\mathcal{A}))$*). Using the isomorphism (and Lemma 6), this means that* $\nu(x*y) = \nu(\nu x*\nu y) \leq \nu x * \nu y \leq \nu(x * y)$ *(for every* x *and* y *in* A*).*

Example 28 *Consider the t-norms* $\mathcal{T}_{T,\alpha}$ *from Example 18. As the diagonal of* $\mathcal{L}^I$ *is closed under* $\mathcal{T}_{T,\alpha}$ *and* $\mathcal{I}_{\mathcal{T}_{T,\alpha}}$*, Theorem 26 implies that* $(L^I, \sqcap, \sqcup, \mathcal{T}_{T,\alpha}, \mathcal{I}_{\mathcal{T}_{T,\alpha}}, \nu, \mu, [0, 0], [0, 1], [1, 1])$ *is a triangle algebra.*

Just like on the unit interval, we can study particular subclasses of residuated t-norms on $\mathcal{L}^I$, and therefore, considering Theorem 26, also particular triangle algebras (provided that the diagonal is closed under the t-norms and their residual implicators).

To this end, recall (e.g. [15,21]) that a t-norm $\mathcal{T}$ on $(L, \sqcap, \sqcup, 0, 1)$ is called divisible if $\mathcal{T}(x, \mathcal{I}_{\mathcal{T}}(x, y)) = x \sqcap y$ and involutive if $\mathcal{I}_{\mathcal{T}}(\mathcal{I}_{\mathcal{T}}(x, 0), 0) = x$ for x, y in L, that a BL-algebra is a divisible prelinear residuated lattice, and that an MV-algebra is an involutive BL-algebra. On the unit interval, a t-norm induces a BL-algebra iff it is continuous, and an MV-algebra iff it is isomorphic to the Łukasiewicz t-norm T_W defined by $T_W(x, y) = \max(0, x + y - 1)$ for x, y in $[0, 1]$. On $\mathcal{L}^{\mathcal{I}}$, neither a BL-algebra nor an MV-algebra exists (as they are subclasses of MTL-algebras), yet in [31], it was proven that, for a t-norm T on $[0, 1]$:

- T is divisible iff for each α in [0,1], $\mathcal{T}_{T,\alpha}$ is weakly divisible, that is, for x, y in L^I,
$$\mathcal{T}(x, \mathcal{I}_{\mathcal{T}}(x, y)) \bigsqcup \mathcal{T}(y, \mathcal{I}_{\mathcal{T}}(y, x)) = x \bigsqcap y$$
- T is involutive iff $\mathcal{T}_{T,0}$ is involutive, hence iff the pseudo t-representable t-norm corresponding to T is involutive; for $\alpha > 0$, $\mathcal{T}_{T,\alpha}$ is never involutive (this is a consequence of Proposition 15, because $\mathcal{T}_{T,\alpha}([0, 1], [0, 1]) = [0, \alpha]$).
- $([0, 1], \min, \max, T, I_T, 0, 1)$ is an MV-algebra iff $(L^I, \bigsqcap, \bigsqcup, \mathcal{T}_{T,0}, \mathcal{I}_{\mathcal{T}_{T,0}}, [0, 0], [1, 1])$ is an involutive, weakly divisible residuated lattice.

BL-algebras and MV-algebras are quintessential in formal fuzzy logics as the algebraic counterparts to Basic Logic BL and Łukasiewicz logic Ł (see Section 1.2). The above results suggest that, in refining the conditions of triangle algebras (which play the same role for $\mathcal{L}^I$ as MTL-algebras do for $([0, 1], \min, \max)$, i.e., they characterize the residuated t-norms) to obtain more powerful structures, we should replace divisibility by weak divisibility.

Note also that the t-norm $\mathcal{T}_{T_W,0}$ on $\mathcal{L}^I$, which seems to satisfy the most useful properties (residuated, weakly divisible, involutive) is not t-representable. At this point, it remains an open question whether every weakly divisible, involutive triangle algebra on $\mathcal{L}^I$ is isomorphic to the triangle algebra induced by $\mathcal{T}_{T_W,0}$.

Another open question is whether every triangle algebra on $\mathcal{L}^I$ is induced by a t-norm of the form $\mathcal{T}_{T,\alpha}$. This might seem unlikely, but until now the only known residuated t-norms $\mathcal{T}$ on $\mathcal{L}^I$ that give it the structure of an IVRL are of this form. If the answer is positive, the answer to the previous open question will be positive too.

2.3 Connections with Truth Stressers, Modal Operators and Rough Sets

In this section, we will examine the relationship between triangle algebras and previously introduced structures:

- Belohlávek and Vychodil [2] defined a so-called "truth stresser" ν for a residuated lattice $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ as a unary operator on L that satisfies T.1, T.5 and T.8.

- Ono [27] defined a modal residuated lattice as a structure $(L, \sqcap, \sqcup, *, \Rightarrow, \nu, 0, 1)$, in which $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice and ν a unary operator on L that satisfies T.1, T.2, T.5, and, for all x and y in L, $\nu(x \sqcap y) \leq \nu x$ and $\nu x * \nu y \leq \nu(x * y)$. From Lemma 6, it follows that the latter two properties are in this case equivalent to T.8. Hence, in a modal residuated lattice, ν is a truth stresser additionally satisfying T.2.
- A Hájek [21] truth stresser for a residuated lattice $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a unary operator ν on L that satisfies T.1, T.2, T.4 [6], T.5, T.8 and $\nu x \sqcup \neg \nu x = 1$ (weakened law of excluded middle, WLEM) for every x and y in L. Hence, $(L, \sqcap, \sqcup, *, \Rightarrow, \nu, 0, 1)$ is a modal residuated lattice in which T.4 and WLEM are satisfied.

  Triangle algebras do not maintain WLEM as, in many cases, it would imply that $\nu x = 0$ if $x \neq 1$. For example, if 1 is $\sqcup$-irreducible, WLEM is equivalent to "$\nu x = 0$ if $x \neq 1$". Only triangle algebras with 3 (or 1) elements can satisfy this property.
- Ciabattoni, Metcalfe and Montagna [5] defined an interior operator on a residuated lattice as a unary operator that satisfies T.1, T.2, T.3, T.5 and, for all x and y in L, $\nu(\nu x * \nu y) = \nu x * \nu y$ and $\nu(x \sqcup y) = \nu(\nu x \sqcup \nu y)$. The latter property is, in this case, weaker than T.4, because we can prove it is fulfilled whenever T.4 is: $\nu(x \sqcup y) = \nu x \sqcup \nu y = \nu\nu x \sqcup \nu\nu y = \nu(\nu x \sqcup \nu y)$. Also in this case, $\nu(\nu x * \nu y) = \nu x * \nu y$ is equivalent to T.8. To prove this, recall from Lemma 6 that T.8 is in this case (because of T.3) equivalent to $\nu x * \nu y \leq \nu(x * y)$. Suppose this property is valid. Then we find, using T.1 and T.2: $\nu(\nu x * \nu y) \leq \nu x * \nu y = \nu\nu x * \nu\nu y \leq \nu(\nu x * \nu y)$. Conversely, we derive T.8 from $\nu(\nu x * \nu y) = \nu x * \nu y$, using T.1 and T.3: $\nu x * \nu y = \nu(\nu x * \nu y) = \nu((\nu x * \nu y) \sqcap (x * y)) = \nu(\nu x * \nu y) \sqcap \nu(x * y) = (\nu x * \nu y) \sqcap \nu(x * y)$, and hence $\nu x * \nu y \leq \nu(x * y)$, which was equivalent to T.8.

  Hence, a residuated lattice with interior operator is a modal residuated lattice additionally satisfying T.3 and $\nu(x \sqcup y) = \nu(\nu x \sqcup \nu y)$.
- A modal operator according to Rachunek and Salounová [28] on a residuated lattice $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a unary operator μ that satisfies T.1', T.2' and $\mu(x * y) = \mu x * \mu y$ for all x and y in L. This last property is in general not true in triangle algebras (see Proposition 14).

We adopted the notations ν and μ from Cattaneo and Ciucci [4], who defined these operators on so-called weak Brouwer de Morgan lattices (wBD lattices). A wBD lattice $(L, \sqcap, \sqcup, ', ^{\sim}, 0, 1)$ is a bounded distributive lattice $(L, \sqcap, \sqcup)$ equipped with two complementations:

- a de Morgan complementation $'$, which is defined as an involutive unary

[6] Actually, Hájek imposed the condition $\nu(x \sqcup y) \leq \nu x \sqcup \nu y$, which is in this case equivalent to T.4, because ν is an increasing operator (due to T.5 and T.8).

operator on L that satisfies $(x \sqcup y)' = x' \sqcap y'$, for all x and y in L, and

- a weak Brouwer complementation $\sim$, which is defined as a unary operator satisfying $x \leq x^{\sim\sim}$ and $(x \sqcup y)^{\sim} = x^{\sim} \sqcap y^{\sim}$ for all x and y in L,

for which $x^{\sim\prime} = x^{\sim\sim}$ (interconnection rule).
They defined νx as $x'^{\sim}$ and μx as $x^{\sim\prime}$.

Proposition 29 *In a wBD lattice* $(L, \sqcap, \sqcup, ', {}^{\sim}, 0, 1)$*, T.1, T.1', T.2, T.2', T.3, T.4', T.5, T.5', T.7 and T.7' are always fulfilled, as well as* $\mu x = (\nu x')'$ *and the de Morgan laws for* $'$.

PROOF. The validity of the properties T.1, T.1', T.2, T.2', T.7 and T.7' is a result of Cattaneo and Ciucci [4]. Since $(x \sqcup y)' = x' \sqcap y'$ holds, also $(x' \sqcup y')'' = (x'' \sqcap y'')'$. Because $'$ is involutive, this means $x' \sqcup y' = (x \sqcap y)'$. So the de Morgan laws are valid. We will use them to prove the remaining properties:

- $\nu(x \sqcap y) = (x \sqcap y)'^{\sim} = (x' \sqcup y')^{\sim} = x'^{\sim} \sqcap y'^{\sim} = \nu x \sqcap \nu y$, so T.3 holds.
- $\mu(x \sqcup y) = (x \sqcup y)^{\sim\prime} = (x^{\sim} \sqcap y^{\sim})' = x^{\sim\prime} \sqcup y^{\sim\prime} = \mu x \sqcup \nu y$, so T.4' holds.
- The complementations $'$ and ${}^{\sim}$ are decreasing operators. We show this for ${}^{\sim}$: if $x \leq y$, then $y^{\sim} = (x \sqcup y)^{\sim} = x^{\sim} \sqcap y^{\sim}$, so $y^{\sim} \leq x^{\sim}$. Because $0 \leq 1^{\sim}$, we find $1 \leq 1^{\sim\sim} \leq 0^{\sim}$. Therefore $0^{\sim} = 1$. Similarly $0' = 1$ and, by involutivity $1' = 0'' = 0$. So we can now prove that $\nu 1 = 1'^{\sim} = 0^{\sim} = 1$ and $\mu 0 = 0^{\sim\prime} = 1' = 0$. Hence T.5 and T.5' hold.
- $(\nu x')' = (x''^{\sim})' = (x^{\sim})' = \mu x$.

□

Note that T.3' and T.4 are not always satisfied, because $(x \sqcap y)^{\sim}$ is not necessarily equal to $x^{\sim} \sqcup y^{\sim}$.
Some triangle algebras can be seen as wBD lattices:

Proposition 30 If $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is a distributive triangle algebra, if $'$ is a de Morgan complementation on $(A, \sqcap, \sqcup, 0, 1)$ such that $\mu x = (\nu x')'$ and if we define ${}^{\sim}$ by $x^{\sim} = (\mu x)'$, then $(A, \sqcap, \sqcup, ', {}^{\sim}, 0, 1)$ is a wBD lattice.

PROOF. We have to prove that ${}^{\sim}$ is a weak Brouwer complementation and that $x^{\sim\prime} = x^{\sim\sim}$, for all x in A.
First note that, because $\mu x = (\nu x')'$ for every x in A and $'$ is involutive, $(\mu x')' = (\nu x'')'' = \nu x$.
Furthermore, using T.7 we find $\mu x = \nu \mu x = (\mu((\mu x)'))' = ((\mu x)')^{\sim} = x^{\sim\sim}$.

- Using T.1', we obtain $x \leq \mu x = x^{\sim\sim}$.
- Applying T.4', we find $(x \sqcup y)^{\sim} = (\mu(x \sqcup y))' = (\mu x \sqcup \mu y)' = (\mu x)' \sqcap (\mu y)' = x^{\sim} \sqcap y^{\sim}$.
- Finally $x^{\sim\prime} = (\mu x)'' = \mu x = x^{\sim\sim}$.

$\square$

Finally, it can be seen that a triangle algebra $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ induces a rough approximation space $\mathcal{R} = (A, E(\mathcal{A}), \nu, \mu)$ (in the sense of Cattaneo [3]) in which

- A is the set of approximable elements,
- $E(\mathcal{A})$ is the set of exact or 'definable' elements,
- ν: $A \to E(\mathcal{A})$ is the inner approximation map, satisfying $(\forall x \in E(\mathcal{A}))(\forall y \in A)(x \leq y \text{ iff } x \leq \nu y)$,
- μ: $A \to E(\mathcal{A})$ is the outer approximation map, satisfying $(\forall x \in A)(\forall y \in E(\mathcal{A}))(x \leq y \text{ iff } \mu x \leq y)$,

and in which for any element x in A, its rough approximation is defined by $(\nu x, \mu x)$. The proof is given below. In this case, T.9 ensures that no two different elements have the same rough approximation.

PROOF. Suppose $x \in E(\mathcal{A})$ and $y \in A$. If $x \leq y$, then $x = \nu x \leq \nu y$ (since ν is increasing). Conversely, if $x \leq \nu y$, then $x \leq \nu y \leq y$ (using T.1).
Suppose $x \in A$ and $y \in E(\mathcal{A})$. If $x \leq y$, then $\mu x \leq \mu y = y$ (since μ is increasing). Conversely, if $\mu x \leq y$, then $x \leq \mu x \leq y$ (using T.1'). $\square$

3 Triangle logic (TL)

In this section we translate the defining properties of triangle algebras into logical axioms, and show that the resulting logic TL is sound and complete w.r.t. the variety of triangle algebras.

The language of TL consists of countably many proposition variables (p_1, p_2,...), the constants $\overline{0}$ and $\overline{u}$, the unary operators $\Box$, $\Diamond$, the binary operators $\wedge$, $\vee$, &, $\rightarrow$, and finally the auxiliary symbols '(' and ')'. Formulas are defined inductively: proposition variables, $\overline{0}$ and $\overline{u}$ are formulas; if ϕ and ψ are formulas, then so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \& \psi)$, $(\phi \rightarrow \psi)$, $(\Box \psi)$ and $(\Diamond \psi)$.

In order to avoid unnecessary brackets, we agree on the following priority rules:

- unary operators always take precedence over binary ones, while

- among the binary operators, & has the highest priority; furthermore $\wedge$ and $\vee$ take precedence over $\rightarrow$,
- the outermost brackets are not written.

We also introduce some useful shorthand notations: $\overline{1}$ for $\overline{0} \rightarrow \overline{0}$, $\neg\phi$ for $\phi \rightarrow \overline{0}$ and $\phi \leftrightarrow \psi$ for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$ for formulas ϕ and ψ..

The axioms of TL are those of ML (Monoidal Logic) [23], i.e.

$$
\begin{array}{ll}
ML.1 & (\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi)),\\
ML.2 & \phi \rightarrow (\phi \vee \psi),\\
ML.3 & \psi \rightarrow (\phi \vee \psi),\\
ML.4 & (\phi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\phi \vee \psi) \rightarrow \chi)),\\
ML.5 & (\phi \wedge \psi) \rightarrow \phi,\\
ML.6 & (\phi \wedge \psi) \rightarrow \psi,\\
ML.7 & (\phi \& \psi) \rightarrow \phi,\\
ML.8 & (\phi \& \psi) \rightarrow (\psi \& \phi),\\
ML.9 & (\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow \chi) \rightarrow (\phi \rightarrow (\psi \wedge \chi))),\\
ML.10 & (\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \& \psi) \rightarrow \chi),\\
ML.11 & ((\phi \& \psi) \rightarrow \chi) \rightarrow (\phi \rightarrow (\psi \rightarrow \chi)),\\
ML.12 & \overline{0} \rightarrow \phi,
\end{array}
$$

complemented with

$$
\begin{array}{llll}
TL.1 & \Box\phi \rightarrow \phi, & TL.1' & \phi \rightarrow \Diamond\phi,\\
TL.2 & \Box\phi \rightarrow \Box\Box\phi, & TL.2' & \Diamond\Diamond\phi \rightarrow \Diamond\phi,\\
TL.3 & (\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi), & TL.3' & (\Diamond\phi \wedge \Diamond\psi) \rightarrow \Diamond(\phi \wedge \psi),\\
TL.4 & \Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi), & TL.4' & \Diamond(\phi \vee \psi) \rightarrow (\Diamond\phi \vee \Diamond\psi),\\
TL.5 & \Box\overline{1}, & TL.5' & \neg\Diamond\overline{0},\\
TL.6 & \neg\Box\overline{u}, & TL.6' & \Diamond\overline{u},\\
TL.7 & \Diamond\phi \rightarrow \Box\Diamond\phi, & TL.7' & \Diamond\Box\phi \rightarrow \Box\phi,\\
TL.8 & \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi), & & \\
TL.9 & (\Box\phi \leftrightarrow \Box\psi)\&(\Diamond\phi \leftrightarrow \Diamond\psi) \rightarrow (\phi \leftrightarrow \psi), & & \\
TL.10 & (\Box x \rightarrow \Box y) \rightarrow \Box(\Box x \rightarrow \Box y). & &
\end{array}
$$

The deduction rules are modus ponens (MP, from ϕ and $\phi \to \psi$ infer ψ), generalization [7] (G, from ϕ infer $\Box\phi$) and monotonicity of $\Diamond$ (M$\Diamond$, from $\phi \to \psi$ infer $\Diamond\phi \to \Diamond\psi$).

The consequence relation $\vdash$ is defined as follows, in the usual way. Let V be a theory, i.e., a set of formulas in TL. A (formal) proof of a formula ϕ in V is a finite sequence of formulas with ϕ at its end, such that every formula in the sequence is either an axiom of TL, a formula of V, or the result of an application of an inference rule to previous formulas in the sequence. If a proof for ϕ exists in V, we say that ϕ can be deduced from V and we denote this by $V \vdash \phi$.
For a theory V, and formulas ϕ and ψ in TL, denote $\phi \sim_V \psi$ iff $V \vdash \phi \to \psi$ and $V \vdash \psi \to \phi$ (this is also equivalent with $V \vdash \phi \leftrightarrow \psi$).

Note that TL.5 is in fact superfluous, as it immediately follows from $\emptyset \vdash \overline{1}$ and generalization; we include it here to obtain full correspondence with Definition 3.

Definition 31 *Let $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra and V a theory. An $\mathcal{A}$-evaluation is a mapping e from the set of formulas of TL to A that satisfies, for each two formulas ϕ and ψ: $e(\phi \wedge \psi) = e(\phi) \sqcap e(\psi)$, $e(\phi \vee \psi) = e(\phi) \sqcup e(\psi)$, $e(\phi \& \psi) = e(\phi) * e(\psi)$, $e(\phi \to \psi) = e(\phi) \Rightarrow e(\psi)$, $e(\Box\phi) = \nu e(\phi)$, $e(\Diamond\phi) = \mu e(\phi)$, $e(\overline{0}) = 0$ and $e(\overline{u}) = u$. If an $\mathcal{A}$-evaluation e satisfies $e(\chi) = 1$ for every χ in V, it is called an $\mathcal{A}$-model for V.*

It is easy to check that Triangle Logic is sound w.r.t. the variety of triangle algebras, i.e., that if a formula ϕ can be deduced from a theory V in TL, then for every triangle algebra $\mathcal{A}$ and for every $\mathcal{A}$-model e of V, $e(\phi) = 1$. Indeed, we need to verify the soundness of the new axioms and deduction rules of TL (for the axioms and rules of ML, the proofs (in ML) can be copied). For the axioms this is easy, as they are straightforward generalizations of axioms of triangle algebras. We will now verify the soundness of the new deduction rules.

- Generalization. We need to prove: if for all triangle algebras $\mathcal{A}$ and all $\mathcal{A}$-models e for V, $e(\phi) = 1$, then for all triangle algebras $\mathcal{A}$ and all $\mathcal{A}$-models e for V, $e(\Box\phi) = 1$. Take such a triangle algebra and such a model e. Then $e(\phi) = 1$, and $e(\Box\phi) = \nu e(\phi) = \nu 1 = 1$.
- Monotonicity of $\Diamond$. We need to prove: if for all triangle algebras $\mathcal{A}$ and all $\mathcal{A}$-models e for V, $e(\phi \to \psi) = 1$, then for all triangle algebras $\mathcal{A}$ and all $\mathcal{A}$-models e for V, $e(\Diamond\phi \to \Diamond\psi) = 1$. Take such a triangle algebra and model e. Then $e(\phi) \Rightarrow e(\psi) = e(\phi \to \psi) = 1$, which means $e(\phi) \le e(\psi)$. Then $e(\Diamond\phi) = \mu e(\phi) \le \mu e(\psi) = e(\Diamond\psi)$ because μ is increasing, and thus

[7] Generalization is often called necessitation, e.g. in [33]

$e(\Diamond\phi \to \Diamond\psi) = e(\Diamond\phi) \Rightarrow e(\Diamond\psi) = 1.$

To show that TL is also complete (i.e. that the converse of soundness also holds), we will apply a general result from abstract algebraic logic (shortly AAL, see e.g. [17] for a survey). We start by showing that Triangle Logic is an implicative logic (in the sense of Rasiowa [29]). An implicative logic is a logic (defined by its logical language and consequence relation) in which a connective $\to$ exists in the logical language that satisfies:

- $V \vdash \phi \to \phi$,
- $V \vdash \phi \to \chi$ if $V \vdash \phi \to \psi$ and $V \vdash \psi \to \chi$,
- $V \vdash \psi$ if $V \vdash \phi \to \psi$ and $V \vdash \phi$,
- $V \vdash \psi \to \phi$ if $V \vdash \phi$, and
- $\sim_V$ is a congruence w.r.t. every connective in the logical language,

for any theory V and for any formulas ϕ, ψ and χ.
Most of these properties hold trivially for TL as they hold for ML (and the axioms of TL include those of ML). We only need to prove that $\sim_V$ is also a congruence w.r.t. $\Box$ and $\Diamond$. Indeed, if $V \vdash \phi \to \psi$, then by M$\Diamond$ $V \vdash \Diamond\phi \to \Diamond\psi$. So if $\phi \sim_V \psi$, then $\Diamond\phi \sim_V \Diamond\psi$. Moreover, if $V \vdash \phi \to \psi$, then also $V \vdash \Box(\phi \to \psi)$ (generalization). Using TL.8 and MP yields $V \vdash \Box\phi \to \Box\psi$. Therefore, if $\phi \sim_V \psi$, then $\Box\phi \sim_V \Box\psi$.
The general result (which is explained in e.g. [16]) we can apply now, is that Triangle Logic is complete w.r.t. the variety of triangle algebras if it is sound w.r.t. it and if in triangle algebras $x = y$ if $x \Rightarrow y = 1$ and $y \Rightarrow x = 1$. Triangle algebras indeed satisfy these conditions, so we can state:

Theorem 32 (**Soundness and completeness of TL**) A formula ϕ can be deduced from a theory V in TL iff for every triangle algebra $\mathcal{A}$ and for every $\mathcal{A}$-model e of V, $e(\phi) = 1$.

Theorem 32 implies similar results for more specific logics.

- For example, if we add $\mu(x * y) = \mu x * \mu y$ (see Proposition 14) to the conditions of a triangle algebra and $\Diamond\phi \& \Diamond\psi \to \Diamond(\phi \& \psi)$ to the axioms of TL, then we can obtain a valid theorem by replacing 'triangle algebra' and 'Triangle Logic' in the formulation of Theorem 32 by the new algebra and logic. This property implies that (in terms of IVRL) the second component of $[x_1, x_2] * [y_1, y_2]$ is independent of x_1 and y_1 (in a similar way as in Remark 27). This means that we can use this property to characterize IVRLs with t-representable t-norms by triangle algebras satisfying $\mu(x * y) = \mu x * \mu y$.
- Another interesting example is $x = \neg\neg x$, connected to the axiom $\neg\neg\phi \to \phi$. At the end of Section 2.2 we recalled that the only involutive t-norms of the form (3) were the pseudo t-representable ones. More generally, in an involutive triangle algebra, $u * u = 0$ (see Proposition 15).
- A third example of a condition that could be added is weak divisibility:

$(x * (x \Rightarrow y)) \sqcup (y * (y \Rightarrow x)) = x \sqcap y$. As a logical axiom, this becomes $(\phi \wedge \psi) \to ((\phi \& (\phi \to \psi)) \vee (\psi \& (\psi \to \phi)))$. In Section 2.2, we saw that, for the considered class of t-norms, this property is equivalent with the divisibility property on the diagonal. It is an open problem if in general this is still true. Formally: is weak divisibility equivalent with $\nu x * (\nu x \Rightarrow \nu y) = \nu x \sqcap \nu y$ in triangle algebras?
- As a final example, we can add $(\nu x \Rightarrow \nu y) \sqcup (\nu y \Rightarrow \nu x) = 1$ to the conditions of a triangle algebra (remark that this property is always satisfied for triangle algebras on $\mathcal{L}^I$, because its diagonal is linearly ordered). If also we add $(\Box\phi \to \Box\psi) \vee (\Box\psi \to \Box\phi)$ to the axioms of TL, then again we obtain a valid theorem by replacing 'triangle algebra' and 'Triangle Logic' in Theorem 32 by the new algebra and logic. In this case $(E(\mathcal{A}), \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is an MTL-algebra (prelinear residuated lattice). This means that it is a subalgebra of the direct product of a system of linearly ordered residuated lattices [23]. Using this property, a stronger form of completeness, called chain completeness in [15], can be proven for MTL: a formula ϕ can be deduced from a theory V in MTL iff for every linearly ordered MTL-algebra $\mathcal{A}$ and for every $\mathcal{A}$-model e of V, $e(\phi) = 1$. Similar results hold for subvarieties of the variety of MTL-algebras and their corresponding logics (e.g. BL and Ł). We would like to find analogous theorems for triangle algebras (and subvarieties) too, but at this moment it is still an open question if every triangle algebra satisfying $(\nu x \Rightarrow \nu y) \sqcup (\nu y \Rightarrow \nu x) = 1$ is a subalgebra of the direct product of a system of triangle algebras with linearly ordered diagonal.

Remark 33 *Triangle Logic is a truth-functional logic: the truth degree of a compound proposition is determined by the truth degree of its parts. This causes some counterintuitive results, if we want to interpret the element* $[0, 1]$ *of an IVRL as uncertainty. For example: suppose we don't know anything about the truth value of propositions p and q, i.e.,* $v(p) = v(q) = [0, 1]$. *Then yet the implication* $p \to q$ *is definitely valid:* $v(p \to q) = v(p) \Rightarrow v(q) = [1, 1]$. *However, if* $\neg[0, 1] = [0, 1]$ [8] *(which is intuitively preferable, since the negation of an uncertain proposition is still uncertain), then we can take* $q = \neg p$, *and obtain that* $p \to \neg p$ *is true. Or, equivalently (using the residuation principle), that* $p \& p$ *is false. This does not seem intuitive, as one would rather expect* $p \& p$ *to be uncertain if p is uncertain.*
Another consequence of $[0, 1] \Rightarrow [0, 1] = [1, 1]$ *is that it is impossible to interpret the intervals as a set in which the 'real' (unknown) truth value is contained, and* $X \Rightarrow Y$ *as the smallest closed interval containing every* $x \Rightarrow y$, *with x in X and y in Y (as in [13]). Indeed:* $1 \in [0, 1]$ *and* $0 \in [0, 1]$, *but* $1 \Rightarrow 0 = 0 \notin [1, 1]$.
On the other hand, for t-norms it is possible that $X * Y$ *is the smallest closed*

[8] This is for example the case if $\neg$ is involutive.

interval containing every $x * y$, with x in X and y in Y, but only if they are t-representable (described by the axiom $\mu(x * y) = \mu x * \mu y$). However, in this case $\neg[0, 1] = [0, 0]$, which does not seem intuitive ('the negation of an uncertain proposition is absolutely false').
These considerations seem to suggest that Triangle Logic is not suitable to reason with uncertainty. This does not mean that intervals are not a good way for representing degrees of uncertainty, only that they are not suitable as truth values in a truth functional logical calculus when we interpret them as expressing uncertainty. It might even be impossible to model uncertainty as a truth value in any truth-functional logic. This question is discussed in [11,12]. However, nothing prevents the intervals in Triangle Logic from having more adequate interpretations.

4 Conclusion and Future Work

We have introduced the notion of triangle algebra, a new algebraic structure that models the triangular structure of the set of closed intervals of a residuated lattice. Using an isomorphism, we formally proved the equivalence between triangle algebras and interval-valued residuated lattices. The new structure has connections with, amongst others, rough sets and modal logic. We constructed the corresponding Triangle Logic and proved that this logic is sound and complete with respect to triangle algebras. These results also hold for axiomatic extensions of Triangle Logic and their corresponding subvarieties.
Hopefully this first step will allow us to prove a stronger result, namely soundness and completeness of Triangle Logic (extended with the axiom $(\Box\phi \rightarrow \Box\psi)\vee(\Box\psi \rightarrow \Box\phi)$) with respect to triangle algebras with linearly ordered diagonal. Or even stronger: with respect to triangle algebras on $\mathcal{L}^I$ ('standard' completeness).
Another challenge is to find out if these standard triangle algebras are all determined by t-norms from the general class defined by Formula (3).

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