Fuzzy Region Connection Calculus: An Interpretation Based on Closeness

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Abstract

One of the key strengths of the region connection calculus (RCC) — its generality — is also one of its most important drawbacks for practical applications. The semantics of all the topological relations of the RCC are based on an interpretation of connection between regions. Because of the manner in which the spatial relations are defined, given a particular interpretation of connection, the RCC relations are often hard to evaluate, and their semantics difficult to grasp. Our generalization of the RCC, in which the spatial relations can be fuzzy relations, inherits this limitation of the RCC. To cope with this, in this paper, we provide specific characterizations of the fuzzy spatial relations, corresponding to the particular case where connection is defined in terms of closeness between fuzzy sets. These characterizations pave the way for practical applications in which the notion of connection is graded rather than black-and-white.

Key words: Region Connection Calculus, Fuzzy Relational Calculus, Approximate Equality

1 Introduction

The region connection calculus [3] is one of the best-known and most widely used formalisms for qualitative reasoning about space. Although only topological information can be expressed in the RCC (e.g., A is a part of B, A overlaps with B), and not, for example, qualitative information about the size, shape, distance, or orientation of spatial entities, its expressivity has proved sufficiently general for many real-world applications. The starting point is to define topological relations between regions based on a primitive reflexive and symmetric (dyadic) relation C modelling connection between regions. One important feature of the RCC is that it imposes no further restrictions on how

connection should be interpreted or how regions should be represented, thus obtaining a framework appropriate for a wide array of contexts. As can be seen from Table 1, other spatial relations can be defined in terms of C, using a first-order logic representation.

Note how all these qualitative relations are defined without referring to points, i.e., by taking regions, rather than points, as primitive spatial objects. This characteristic makes the RCC essentially the spatial counterpart of Allen's well-known framework for qualitative reasoning about time [2]. However, the definitions of RCC relations like P and O, which involve quantifiers that range over arbitrary regions, are difficult to evaluate. Furthermore, it is often unclear how a specific interpretation of C influences the semantics of relations like P and O. In other words, the generality of the framework — achieved by treating regions as primitive objects, independent of a particular representation — may actually be undesirable in practical applications. Therefore, more intuitive characterizations of the RCC relations, corresponding to a particular interpretation of C and certain assumptions on how regions are defined, are generally used.

In a companion paper [16], we have introduced a generalization of the RCC in which the spatial relations are fuzzy relations, i.e., mappings from $U \times U$ to $[0, 1]$, where U is the universe of all regions. For example, for two regions u and v, $P(u, v)$ expresses the degree to which u is a part of v. Such a fuzzification is useful in many contexts, including applications involving vague geographical regions (e.g., the Alps, Downtown Chicago, Western Europe, etc.) and applications where space is used in a metaphorical way (e.g., where C expresses the similarity between objects). Another context where fuzzy spatial relations are more appropriate than crisp relations, is when the abrupt transition between, for example, EC and DC, or TPP and $NTPP$ is counterintuitive. In many situations, it is impossible or undesirable to differentiate between spatial configurations where two objects are touching each other (i.e., TPP or EC holds), and spatial configurations where the objects are very close to each other, but not touching (i.e., $NTPP$ or DC holds). One solution to this problem is to define two regions u and v to be connected if at least one point of u is *close* to one point of v, where the notion of closeness requires a definition of C as a fuzzy relation. The fuzzy relations obtained in [16] to generalize the original RCC relations, are also shown in Table 1, where T is a left-continuous t-norm and I_T is its residual implicator (see Section 2). We refer to [16] for a motivation for these generalized definitions, their properties, as well as an overview of related approaches.

The aim of this paper is to provide specific definitions of our fuzzy spatial relations corresponding to a particular interpretation of C and a particular way of representing regions. Specifically, we show how our generalized RCC relations can be used to define topological relations between vague regions, represented

Table 1. Definition of topological relations in the RCC; a and b denote regions, i.e., elements of the universe of regions U.

Name	Relation	Original Definition	Generalized Definition
Disconnected From	$DC(a,b)$	$\neg C(a,b)$	$1 - C(a,b)$
Part Of	$P(a,b)$	$(\forall c \in U)(C(c,a) \Rightarrow C(c,b))$	$\inf_{c \in U} I_T(C(c,a), C(c,b))$
Proper Part Of	$PP(a,b)$	$P(a,b) \wedge \neg P(b,a)$	$\min(P(a,b), 1 - P(b,a))$
Equal To	$EQ(a,b)$	$P(a,b) \wedge P(b,a)$	$\min(P(a,b), P(b,a))$
Overlaps With	$O(a,b)$	$(\exists c \in U)(P(c,a) \wedge P(c,b))$	$\sup_{c \in U} T(P(c,a), P(c,b))$
Discrete From	$DR(a,b)$	$\neg O(a,b)$	$1 - O(a,b)$
Partially Overlaps With	$PO(a,b)$	$O(a,b) \wedge \neg P(a,b) \wedge \neg P(b,a)$	$\min(O(a,b), 1 - P(a,b), 1 - P(b,a))$
Externally Connected To	$EC(a,b)$	$C(a,b) \wedge \neg O(a,b)$	$\min(C(a,b), 1 - O(a,b))$
Non-Tangential Part	$NTP(a,b)$	$P(a,b) \wedge \neg(\exists c \in U)(EC(c,a) \wedge EC(c,b))$	$\inf_{c \in U} I_T(C(c,a), O(c,b))$
Tangential PP	$TPP(a,b)$	$PP(a,b) \wedge \neg NTP(a,b)$	$\min(PP(a,b), 1 - NTP(a,b))$
Non-Tangential PP	$NTPP(a,b)$	$PP(a,b) \wedge NTP(a,b)$	$\min(1 - P(b,a), NTP(a,b))$

as fuzzy sets of points, and how a notion of closeness can be incorporated to obtain a gradual transition between EC and DC, and $NTPP$ and TPP. As a side effect, we obtain a model in which both topological relations and (possibly vague) distance relations between regions can be specified. The paper is structured as follows. First, in Section 2, we recall some important preliminaries from fuzzy set theory and mathematical topology. Next, in Section 3, we discuss how closeness between points can be represented as a fuzzy relation between points. In Section 4, we show how regions can be represented as fuzzy sets, and how our model for closeness of points can be leveraged to a model of closeness of regions. Finally, in Section 5, we provide a characterization of the generalized RCC relations for the special case where C is defined using this model of closeness between regions. Some concluding remarks are presented in Section 6. A preliminary version of some of the results in this paper appeared earlier in [14].

2 Preliminaries

2.1 Fuzzy sets

A fuzzy set [1] A in a universe X is defined as a mapping from X to the unit interval $[0, 1]$. For x in X, $A(x)$ is called the membership degree of x in A. If there exists an x in X such that $A(x) = 1$, A is called normalized. A fuzzy set R in $X \times X$ is called a fuzzy relation in X. R is called reflexive iff $R(x, x) = 1$ for all x in X, and symmetric iff $R(x, y) = R(y, x)$ for all x and y in X.

A t-norm T is defined as a symmetric, associative, increasing $[0, 1]^2 - [0, 1]$ mapping satisfying the boundary condition $T(x, 1) = x$ for all x in $[0, 1]$. Some common t-norms are the minimum T_M, the product T_P and the Łukasiewicz t-norm T_W, defined by:

$$\begin{aligned} T_M(x, y) &= \min(x, y) \\ T_P(x, y) &= xy \\ T_W(x, y) &= \max(0, x + y - 1) \end{aligned}$$

The negation of an element x in $[0, 1]$ is commonly defined by $1 - x$. Finally, a $[0, 1]^2 - [0, 1]$ mapping I which is decreasing in the first and increasing in the second argument and which satisfies $I(0, 0) = I(0, 1) = I(1, 1) = 1$ and $I(1, 0) = 0$ is called an implicator.

Let T be an arbitrary t-norm; it can be shown that the mapping I_T, defined for x and y in $[0, 1]$ by:

$$I_T(x, y) = \sup\{\lambda | \lambda \in [0, 1] \text{ and } T(x, \lambda) \leq y\} \tag{1}$$

is an implicator, which is called the residual implicator of T. For example, the residual implicator corresponding to T_W is given by:

$$I_{T_W}(x, y) = \min(1, 1 - x + y)$$

for all x and y in $[0, 1]$. For convenience, we will write I_W instead of I_{T_W} in the remainder of this paper. If T is a left-continuous t-norm (i.e., a t-norm whose partial mappings are left-continuous such as T_M, T_P, and T_W), it can be shown that for all x, y, z and u in $[0, 1]$, J an arbitrary index set and $(x_j)_{j\in J}$ and $(y_j)_{j\in J}$ families in $[0, 1]$, it holds that (see e.g., [8])

$$T(x, y) \leq z \Leftrightarrow x \leq I_T(y, z) \tag{2}$$
$$x \leq y \Leftrightarrow I_T(x, y) = 1 \tag{3}$$
$$T(I_T(x, y), z) \leq I_T(x, T(y, z)) \tag{4}$$
$$I_T(T(x, y), z) = I_T(x, I_T(y, z)) \tag{5}$$
$$T(I_T(x, y), I_T(y, z)) \leq I_T(x, z) \tag{6}$$
$$T(I_T(x, y), I_T(z, u)) \leq I_T(T(x, z), T(y, u)) \tag{7}$$
$$T(\sup_{j\in J} x_j, y) = \sup_{j\in J} T(x_j, y) \tag{8}$$
$$I_T(\sup_{j\in J} x_j, y) = \inf_{j\in J} I_T(x_j, y) \tag{9}$$
$$I_T(x, \inf_{j\in J} y_j) = \inf_{j\in J} I_T(x, y_j) \tag{10}$$
$$T(\inf_{j\in J} x_j, y) \leq \inf_{j\in J} T(x_j, y) \tag{11}$$
$$I_T(x, \sup_{j\in J} y_j) \geq \sup_{j\in J} I_T(x, y_j) \tag{12}$$

Moreover, it is easy to see that for an arbitrary t–norm T it holds that

$$I_T(1, x) = x \tag{13}$$

Note that for any implicator I, it holds that $I(x, 1) \geq I(0, 1) = 1$, for every x in $[0, 1]$. Throughout this paper, we will always assume that T is a left-continuous t-norm.

2.2 Topological interpretations of the RCC

The standard semantics of C are specified in terms of mathematical topology. Therefore, we briefly recall some basic notions from classical (point-set) topology. Let X be a non-empty set and τ a subset of the power set 2^X of X. The set τ is called a topology on X iff

(1) $\emptyset \in \tau$ and $X \in \tau$
(2) $A \in \tau \wedge B \in \tau \Rightarrow A \cap B \in \tau$
(3) $(\forall i \in I)(A_i \in \tau) \Rightarrow \bigcup_{i\in I} A_i \in \tau$

A subset A of X is called open iff $A \in \tau$ and closed if its complement $X \setminus A$ is open. The interior $i(A)$ of A is the largest open set that is contained in A, while the closure $cl(A)$ of A is the smallest closed set that contains A. Finally, A is called regular open iff $i(cl(A)) = A$ and regular closed iff $cl(i(A)) = A$.

Usually, in the RCC, regions are assumed to be regular closed sets, and two regions are said to be connected if they share at least one point [6]. In this interpretation, P corresponds to the subset relation, while O holds between two regions if their interiors share at least one point. Another possibility is to define regions as regular open sets, and to define two regions to be connected if their closures share at least one point. In this case, for example, O holds between two regions if they share at least one point.

3 Modelling closeness between points

A natural way to model closeness between points is to use models for approximate equality. In particular, fuzzy T-equivalence relations seem to be an appropriate candidate, at first glance. Recall that a fuzzy T-equivalence relation (w.r.t. a t-norm T) in a universe X is a reflexive, symmetric fuzzy relation R in X that satisfies T-transitivity, that is

$$T(R(x, y), R(y, z)) \leq R(x, z)$$

for all x, y, and z in X. However, using fuzzy T-equivalence relations imposes rather strict limitations on the interpretation of approximate equality, and therefore closeness. Problems occur in situations where we want to define two points to be close to degree 1, even if their distance is strictly positive. For example, consider a two-dimensional Euclidean space, and assume that, whenever the distance between two points is less than or equal to 0.1, we call these points close to degree 1. If we have three points a, b, and c such that $d(a, b) = 0.1$, $d(b, c) = 0.1$ and $d(a, c) = 0.2$ (i.e., a, b, and c are on a line), then a and b are close to degree 1, b and c are close to degree 1, by definition. If we impose T-transitivity on the closeness relation, a and c have to be close to degree 1 as well. Since it is natural to define closeness (in a given context) only in terms of the distance between two points, this means that any two points whose distance is less than 0.2, are close to degree 1. Repeating this argument, we obtain that any two points whose distance is less than 0.4, 0.8, 1.6, etc. are close to degree 1.

To avoid such problems, we will use the more general notion of a resemblance relation [11,12]. Recall that a mapping d from X^2 to $[0, +\infty[$ is called a pseudometric on X iff $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, y) + d(y, z) \geq d(x, z)$ for all x, y and z in X. A fuzzy relation R in X is called a resemblance relation

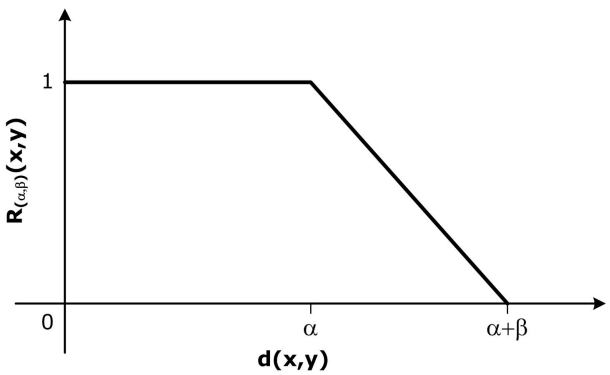


Fig. 1. Resemblance relation R.

w.r.t. a pseudometric d on X iff for all x, y, z and u in X

$$R(x,x) = 1 \tag{14}$$

$$d(x,y) \leq d(z,u) \Rightarrow R(x,y) \geq R(z,u) \tag{15}$$

Note that (15) implies that any resemblance relation is also symmetric. However, the third property of fuzzy T-equivalence relations, T-transitivity, does not hold anymore in general.

For example, let $\alpha \geq 0$, $\beta \geq 0$, and let d be a pseudometric on X. The fuzzy relation $R_{(\alpha,\beta)}$ in X defined for all x and y in X as

$$R_{(\alpha,\beta)}(x,y) = \begin{cases} 1 & \text{if } d(x,y) \leq \alpha \\ 0 & \text{if } d(x,y) > \alpha + \beta \\ \frac{\alpha+\beta-d(x,y)}{\beta} & \text{otherwise} \end{cases} \tag{16}$$

is a resemblance relation w.r.t. d. Figure 1 illustrates the definition of this fuzzy relation in terms of the distance between x and y. Note how the parameter β defines how smooth the transition is from close to not close, while α defines how close two points should be located from each other to be considered definitely close, i.e., close to degree 1. The fact that $R_{(\alpha,\beta)}$ satisfies (15) can be seen from the fact that this graph is decreasing. If $\alpha > 0$, $R_{(\alpha,\beta)}$ is not T-transitive for any t-norm T. To see this, let a, b, and c be collinear points such that $d(a,b) = d(b,c) = \alpha$ and $d(a,c) = 2\alpha$. It holds that $T(R_{(\alpha,\beta)}(a,b), R_{(\alpha,\beta)}(b,c)) = T(1,1) = 1$, while $R_{(\alpha,\beta)}(a,c) < 1$.

The following lemma will be useful to derive specific definitions of the generalized RCC relations in Appendix A.

Lemma 1. *Let $(X, \|.\|)$ be a normed vector space, d the induced metric (i.e., $d(x,y) = \|y - x\|$ for all x and y in X), and R a resemblance relation w.r.t. d. It holds that the fuzzy relation E in X defined for all x and z in X by*

$$E(x,z) = \inf_{y \in X} I_T(R(x,y), R(y,z)) \tag{17}$$

is a fuzzy T-equivalence relation in U.

Proof. The reflexivity of E follows immediately from the symmetry of R and (3). To show the symmetry of E, we use the fact that, since R satisfies (15), there must exist a function f from $[0, +\infty[$ to $[0, 1]$ such that $R(x, y) = f(d(x, y))$ for every x and y in X. We obtain

$$\begin{aligned}
E(x,z) &= \inf_{y\in X} I_T(R(x,y), R(y,z)) \\
&= \inf_{y_0\in X} I_T(R(x, x+z-y_0), R(x+z-y_0, z)) \\
&= \inf_{y_0\in X} I_T(f(d(x, x+z-y_0)), f(d(x+z-y_0, z))) \\
&= \inf_{y_0\in X} I_T(f(\|x+z-y_0-x\|), f(\|z-(x+z-y_0)\|)) \\
&= \inf_{y_0\in X} I_T(f(\|z-y_0\|), f(\|y_0-x\|)) \\
&= \inf_{y_0\in X} I_T(f(d(z,y_0)), f(d(y_0,x))) \\
&= \inf_{y_0\in X} I_T(R(z,y_0), R(y_0,x)) \\
&= E(z,x)
\end{aligned}$$

Finally, the T-transitivity of E follows from (11), the symmetry of R, and (6):

$$\begin{aligned}
T(E(a,b), E(b,c)) &= T(\inf_{y\in X} I_T(R(a,y), R(y,b)), \inf_{y\in X} I_T(R(b,y), R(y,c))) \\
&\leq \inf_{y\in X} T(I_T(R(a,y), R(y,b)), \inf_{y'\in X} I_T(R(b,y'), R(y',c))) \\
&\leq \inf_{y\in X} T(I_T(R(a,y), R(y,b)), I_T(R(b,y), R(y,c))) \\
&= \inf_{y\in X} T(I_T(R(a,y), R(y,b)), I_T(R(y,b), R(y,c))) \\
&\leq \inf_{y\in X} I_T(R(a,y), R(y,c)) \\
&= E(a,c)
\end{aligned}$$

□

Corollary 1. *For x, y, and z in X, it holds that*

$$I_T(R(x,y), R(y,z)) \geq E(x,z) \tag{18}$$
$$T(E(x,y), R(y,z)) \leq R(x,z) \tag{19}$$
$$I_T(E(x,z), R(y,z)) \geq R(x,y) \tag{20}$$

where we used (2) *to obtain* (19) *and* (20).

The previous lemma does not hold in general for an arbitrary reflexive and symmetric fuzzy relation R, as is illustrated by the following counterexample.

Example 1. Assume that R is defined as

$$R(x,y) = \begin{cases} 0 & \text{if } (x = b \wedge y \neq b \wedge y \neq a) \text{ or } (x \neq b \wedge x \neq a \wedge y = b) \\ 1 & \text{otherwise} \end{cases}$$

where $a, b \in X$, and $a \neq b$. Obviously, R is reflexive and symmetric. However,

$$E(a,b) = \inf_{y \in X} I_T(R(a,y), R(y,b)) \leq I_T(R(a,c), R(c,b)) = I_T(1,0) = 0$$

where $c \neq a$ and $c \neq b$, while

$$\begin{aligned} & E(b,a) \\ & = \inf_{y \in X} I_T(R(b,y), R(y,a)) \\ & = \min(\inf_{y \neq a,b} I_T(R(b,y), R(y,a)), I_T(R(b,b), R(b,a)), I_T(R(b,a), R(a,a))) \\ & = \min(\inf_{y \neq a,b} I_T(0, R(y,a)), I_T(1,1), I_T(1,1)) \\ & = 1 \end{aligned}$$

hence E is not symmetric, in general, when R does not satisfy (15).

Note that while T-transitivity is not required, and not even desirable, for $R_{(\alpha,\beta)}$, the T-transitivity of the fuzzy relation E defined in (17) will be needed to derive our characterization of the generalized RCC relations. This is the reason why we only consider resemblance relations to model closeness between points, rather than arbitrary symmetric and reflexive fuzzy relations.

4 Modelling regions as fuzzy sets

In the following, regions are defined as normalized fuzzy sets in the universe X. Henceforth, we will always assume that X is equipped with a norm $\|.\|$, that d is the induced metric, and that R is a resemblance relation w.r.t. d.

First, we recall some important constructs from fuzzy relational calculus. The direct image $R{\uparrow}A$ and the superdirect image $R{\downarrow}A$ of a fuzzy set A in X under a fuzzy relation R in X are the fuzzy sets in X defined by [4]

$$(R{\uparrow}A)(y) = \sup_{x \in X} T(R(x,y), A(x)) \tag{21}$$

$$(R{\downarrow}A)(y) = \inf_{x \in X} I_T(R(x,y), A(x)) \tag{22}$$

for all y in X. For notational convenience, we introduce the following abbre-

viations:

$$
\begin{aligned}
R{\uparrow}{\uparrow}A &= R{\uparrow}(R{\uparrow}A) \\
R{\downarrow}{\downarrow}A &= R{\downarrow}(R{\downarrow}A) \\
R{\uparrow}{\downarrow}A &= R{\uparrow}(R{\downarrow}A) \\
R{\downarrow}{\uparrow}A &= R{\downarrow}(R{\uparrow}A)
\end{aligned}
$$

We will also refer to fuzzy sets like $R{\uparrow}{\uparrow}{\downarrow}A$, which are defined analogously. In [10], it is shown that $R{\downarrow}{\uparrow}A$ and $R{\uparrow}{\downarrow}A$ bear close similarity to the concepts of closure and interior from classical topology. A fuzzy set A is called R-closed iff $R{\downarrow}{\uparrow}A = A$ and R-open iff $R{\uparrow}{\downarrow}A = A$ [10]. The fuzzy set $R{\downarrow}{\uparrow}A$ is sometimes called the R-closure of A.

Direct and superdirect images under a fuzzy relation (not necessarily involving a resemblance relation) have proven useful in many contexts. When R is a resemblance relation, however, we can give a specific interpretation to $R{\uparrow}A$, $R{\downarrow}A$, $R{\downarrow}{\uparrow}A$, and $R{\uparrow}{\downarrow}A$. This is illustrated in Figure 2 for a normalized fuzzy set A in $\mathbb{R}$, and $R = R_{(\alpha,\beta)}$ the resemblance relation defined in (16). Note that we use $\mathbb{R}$ for the ease of depicting the membership functions, while in practice, of course, fuzzy sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ are more commonly used to represent regions. Intuitively, $R{\uparrow}A$ is a fuzzy set that contains all the points that are close to some point of the region A (w.r.t. R), while $R{\downarrow}A$ contains the points that are located in A, but not close to the boundary of A, i.e., the points that are located in the heart of the region. The membership functions of $R{\uparrow}{\downarrow}A$ and $R{\downarrow}{\uparrow}A$ are more similar to the membership function of A than those of $R{\uparrow}A$ and $R{\downarrow}A$. In fact, $R{\downarrow}{\uparrow}A$ and $R{\uparrow}{\downarrow}A$ only differ from A in that steep parts of the membership function of A have become more gentle (depending on the parameter β). For $R{\downarrow}{\uparrow}A$ and $R{\uparrow}{\downarrow}A$ this causes an increase and a decrease in membership degrees respectively.

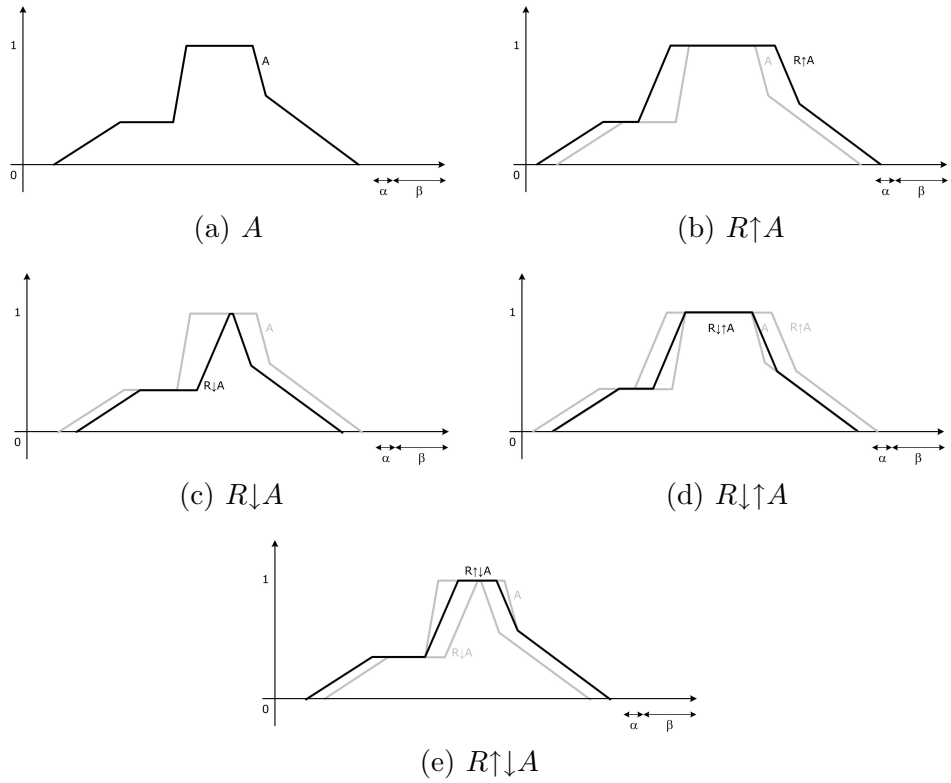
The degree of overlap and the degree of inclusion are frequently used measures to compare two fuzzy sets. The degree of overlap $overl(A, B)$ between two fuzzy sets A and B in X is defined as [4]

$$overl(A, B) = \sup_{x \in X} T(A(x), B(x)) \tag{23}$$

expressing the degree to which there exists an element of X that is contained both in A and in B. In the same way, the degree of inclusion $incl(A, B)$ of A in B is defined as [4]

$$incl(A, B) = \inf_{x \in X} I_T(A(x), B(x)) \tag{24}$$

expressing the degree to which all elements of X that are contained in A, are also contained in B. The degree of overlap and the degree of inclusion both express some graded relationship between two fuzzy sets. Relatedness



(a) A (b) $R{\uparrow}A$

(c) $R{\downarrow}A$ (d) $R{\downarrow}{\uparrow}A$

(e) $R{\uparrow}{\downarrow}A$

Fig. 2. Effect of taking the direct and superdirect image of a fuzzy set A under a resemblance relation $R = R_{(\alpha,\beta)}$.

measures [13] are a more general notion which have (23) and (24) as special cases. In this paper we will use

$$A \circ R \circ B = \sup_{x \in X} T(A(x), \sup_{y \in X} T(R(x,y), B(y))) \tag{25}$$

where A and B are fuzzy sets in X and R is a fuzzy relation in X. In particular when R is a resemblance relation, (25) expresses the degree to which there is an element of A that is located close to an element of B (w.r.t. R). In other words, (25) leverages the closeness of points, defined by the resemblance relation R, to closeness of regions.

5 Characterization of the generalized RCC relations

We define connection of two regions as closeness w.r.t. a resemblance relation R.

Definition 1. *For normalized fuzzy sets A and B in X, we define the degree $C(A,B)$ to which A and B are connected as*

$$C(A,B) = A \circ R \circ B \tag{26}$$

Depending on the context, there may be (at least) two different reasons for introducing closeness in the definition of C. First, we may want to express that small distances should be ignored. Intuitively, C expresses the degree to which A and B have a point in common. However, if A and B have no point in common, but some point of A is very close to some point of B, we still want to have that A is connected to B (to some degree). In other words, the resemblance relation in the definition of C is used to model indiscernibility of locations in this case. The second reason is that we may want to express (vague) distance information. For example, two city neighbourhoods are called connected if they are within walking distance of each other, or within a three kilometer radius, etc. Note that concepts like *within walking distance* can be modelled using the resemblance relation $R_{(\alpha,\beta)}$ by providing suitable values for α and β. To obtain such values, data-driven approaches can be used [15].

When connection between fuzzy sets in X is interpreted as in (26), the definitions in the rightmost column of Table 1 can be used to obtain a corresponding interpretation of the other generalized RCC relations. However, the interpretations of P, O and NTP involve infima and suprema that range over arbitrary regions, i.e., arbitrary normalized fuzzy sets in X. This makes it hard to evaluate, and grasp the meaning of these fuzzy relations under a specific interpretation of C. However, as the following proposition shows, when C is defined as above, the interpretations of P, O and NTP can be characterized in terms of degrees of inclusion and overlap of fuzzy sets. Using these characterizations, the generalized RCC relations can be evaluated much easier, and, moreover, their semantics becomes immediately clear.

Proposition 1. *Let U be the set of all normalized fuzzy sets in X, and let C be defined by* (26)*. It holds that*

$$P(A,B) = incl(R{\downarrow}{\uparrow}A, R{\downarrow}{\uparrow}B) \tag{27}$$

$$O(A,B) = overl(R{\downarrow}{\uparrow}A, R{\downarrow}{\uparrow}B) \tag{28}$$

$$NTP(A,B) = incl(R{\uparrow}A, R{\downarrow}{\uparrow}B) \tag{29}$$

For the proof of this proposition, we refer to Appendix A.

Note that P and O correspond to the usual degree of inclusion and the degree of overlap between the R-closures of the fuzzy sets, while $NTP(A,B)$ is the degree to which every point that is close to a point from A, is contained in the R-closure of B. In other words, $NTP(A,B)$ is the degree to which A is a part of B that is not located close to the boundary of B. When $R = R_{(\alpha,\beta)}$ is used to model closeness, the parameter α can be used to specify, for example, how close two regions should be to be considered connected. This is illustrated in the following example.

Example 2. Consider the normalized fuzzy sets A, B, and D in $\mathbb{R}^2$, defined

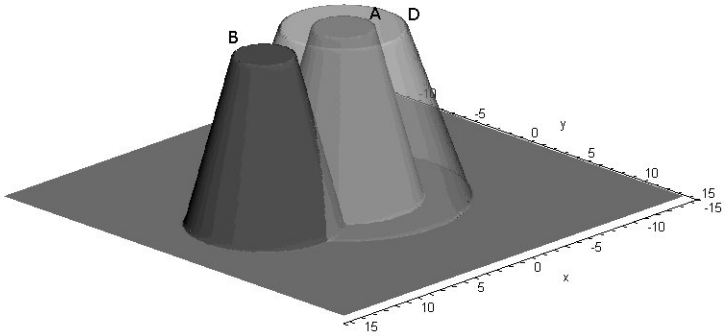


Fig. 3. Normalised fuzzy sets A, B, and C in $\mathbb{R}^2$, representing regions.

for (x, y) in $\mathbb{R}^2$ as

$$A(x,y) = \min(1, \max(0, \frac{5 - \sqrt{x^2 + y^2}}{3}))$$
$$B(x,y) = \min(1, \max(0, \frac{5 - \sqrt{(x-7)^2 + y^2}}{3}))$$
$$D(x,y) = \min(1, \max(0, \frac{8 - \sqrt{x^2 + y^2}}{4}))$$

These fuzzy sets are shown in Figure 3. Using $R_{(1,0)}$ to model closeness and the Łukasiewicz connectives T_W and I_W in the definition of C, it can be shown that

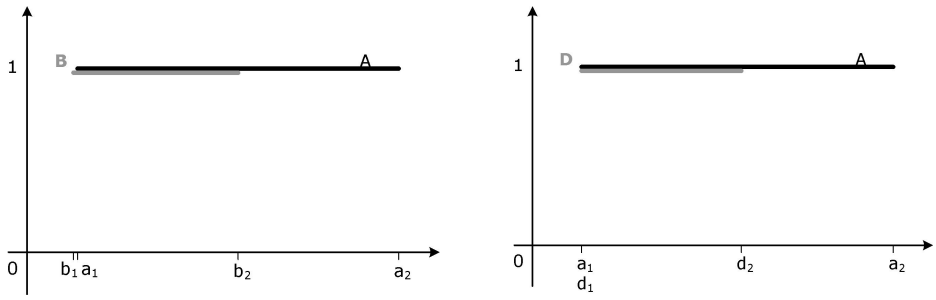
$$O(A,B) = T_W(A(3.5,0), B(3.5,0)) = T_W(0.5, 0.5) = 0$$
$$O(D,B) = T_W(D(5,0), B(5,0)) = T_W(\frac{3}{4}, 1) = \frac{3}{4}$$
$$C(A,B) = T_W(A(4,0), B(5,0)) = T_W(\frac{1}{3}, 1) = \frac{1}{3}$$
$$NTP(A,D) = I_W(A(2,0), D(3,0)) = I_W(1,1) = 1$$

It can indeed be seen from Figure 3 that there is quite some overlap between D and B. On the other hand, the degree of overlap between A and B is too small for $O(A,B) > 0$ to hold. While $O(A,B)$ and $O(D,B)$ are independent of the parameter α, we can obtain different values for $C(A,B)$ and $NTP(A,D)$ by changing α. For example, choosing $\alpha = 2$ yields

$$C(A,B) = T_W(A(3,0), B(5,0)) = T_W(\frac{2}{3}, 1) = \frac{2}{3}$$
$$NTP(A,D) = I_W(A(2,0), D(4,0)) = I_W(1,1) = 1$$

while $\alpha = 3$ leads to

$$C(A,B) = T_W(A(2,0), B(5,0)) = T_W(1,1) = 1$$



(a) $PO(A, B)$ (b) $TPP(D, A)$

Fig. 4. In the usual RCC semantics we have the counterintuitive fact that $PO(A, B)$ and $\neg TPP(B, A)$, while $TPP(D, A)$ and $\neg PO(A, D)$.

$$NTP(A, D) = I_W(A(2, 0), D(5, 0)) = I_W(1, \frac{3}{4}) = \frac{3}{4}$$

Note how increasing the value of α makes the fuzzy relation C more tolerant, and the fuzzy relation NTP less tolerant. For example, A is located somewhat away from the boundary of D, hence $NTP(A, D) = 1$ when α is sufficiently small ($\alpha \leq 2$). However, when α becomes too large (e.g., $\alpha = 3$), A is considered to be too close to the boundary of D for $NTP(A, D) = 1$ to hold. Furthermore, the values of $NTP(A, D)$, $C(A, B)$ and $O(D, B)$ are not independent of each other. In [16] we have provided a transitivity table, which captures such dependencies and can thus be used for fuzzy spatial reasoning. In particular, we have the following transitivity rule:

$$T_W(NTP^{-1}(D, A), C(A, B)) \leq O(D, B)$$

For $\alpha = 3$ we have that $T_W(NTP^{-1}(D, A), C(A, B)) = T_W(\frac{3}{4}, 1) = \frac{3}{4}$, from which we can conclude that $O(D, B) \geq \frac{3}{4}$.

The next example illustrates how appropriate values of the parameter β in $R_{(\alpha,\beta)}$ lead to a gradual transition between generalized RCC relations like PO and TPP.

Example 3. Consider the regions A, B, and D shown in Figure 4, corresponding to the crisp intervals $[a_1, a_2]$, $[b_1, b_2]$, and $[d_1, d_2]$ respectively. Using the original RCC relations, we have that $PO(A, B)$, $\neg TPP(B, A)$, $TPP(D, A)$, and $\neg PO(A, D)$. Nonetheless, the situations depicted in Figure 4(a) and 4(b) are very similar, as the distance between a_1 and b_1 is very small. In many application domains it would be desirable that the spatial relations behave similarly in similar situations. Using our fuzzy relations, this can be achieved because the transition between TPP and PO is gradual for $\beta > 0$. Assume, for example, that $R = R_{(\alpha,\beta)}$ is used, where $\alpha = 5(a_1 - b_1)$, and $\beta = 2(a_1 - b_1)$. It holds that

$$\begin{aligned} &TPP(B, A) \\ &= \min(PP(B, A), 1 - NTP(B, A)) \end{aligned}$$

$$= \min(incl(R{\downarrow}{\uparrow}B, R{\downarrow}{\uparrow}A), 1 - incl(R{\downarrow}{\uparrow}A, R{\downarrow}{\uparrow}B), \\ 1 - incl(R{\uparrow}B, R{\downarrow}{\uparrow}A))$$

When, for example, the Łukasiewicz connectives T_W and I_W are used, we can show that

$$incl(R{\downarrow}{\uparrow}B, R{\downarrow}{\uparrow}A) = 0.5$$
$$incl(R{\downarrow}{\uparrow}A, R{\downarrow}{\uparrow}B) = 0$$
$$incl(R{\uparrow}B, R{\downarrow}{\uparrow}A) = 0$$

Hence, we obtain $TPP(B, A) = 0.5$. In the same way, we can establish that $PO(A, B) = 0.5$, $TPP(D, A) = 1$, and $PO(A, D) = 0$. In this way, we express that although A and B partially overlap to some extent, we could still consider B to be a non-tangential proper part of A as well. Higher values of β correspond to a higher (resp. lower) value of $TPP(B, A)$ (resp. $PO(A, B)$) and vice versa, i.e., the higher the value of β, the more similar the situation in Figure 4(a) is considered to be to the situation in Figure 4(b). For example, when $\beta = 3(a_1 - b_1)$ we have that $TPP(B, A) = 0.66$ and $PO(A, B) = 0.33$. When $\beta \leq a_1 - b_1$ we have that $TPP(B, A) = 0$ and $PO(A, B) = 1$. In other words, the parameter β can be used to control how smooth the transition between, for example, PO and TPP should be.

Finally, we provide two special cases of Proposition 1, corrresponding to situations where the fuzzy sets involved are R-closed, and situations where the resemblance relation R is T-transitive. When A and B are R-closed (i.e., when the membership functions of A and B contain no *steep* parts or discontinuities), we immediately obtain

$$P(A, B) = incl(A, B) \tag{30}$$
$$O(A, B) = overl(A, B) \tag{31}$$
$$NTP(A, B) = incl(R{\uparrow}A, B) \tag{32}$$

If the resemblance relation R is T-transitive, then some of the RCC relations cannot be distinguished anymore. To show this result, we need the following lemma.

Lemma 2. *[10] If R is a fuzzy T-equivalence relation in X, it holds for any fuzzy set A in X that*

$$R{\downarrow}{\uparrow}A = R{\uparrow}A$$

Proposition 2. *If R is T-transitive (i.e., R is a fuzzy T-equivalence relation), it holds that*

$$C(A, B) = O(A, B)$$

Proof. Using Proposition 1 and Lemma 2, we obtain

$$\begin{aligned} O(A,B) &= overl(R{\downarrow}{\uparrow}A, R{\downarrow}{\uparrow}B) \\ &= overl(R{\uparrow}A, R{\uparrow}B) \\ &= \sup_{x\in X} T(\sup_{y\in X} T(R(y,x), A(y)), \sup_{y\in X} T(R(y,x), B(y))) \end{aligned}$$

Using the associativity and symmetry of T, (8), and the symmetry of R, we find

$$\begin{aligned} &= \sup_{x\in X}\sup_{y\in X}\sup_{y'\in X} T(T(R(y,x), A(y)), T(R(y',x), B(y'))) \\ &= \sup_{x\in X}\sup_{y\in X}\sup_{y'\in X} T(A(y), T(T(R(y,x), R(x,y')), B(y'))) \\ &= \sup_{y\in X} T(A(y), \sup_{y'\in X} T(\sup_{x\in X} T(R(y,x), R(x,y')), B(y'))) \end{aligned}$$

and finally, using the T-transitivity of R

$$\begin{aligned} &\leq \sup_{y\in X} T(A(y), \sup_{y'\in X} T(\sup_{x\in X} R(y,y'), B(y'))) \\ &= \sup_{y\in X} T(A(y), \sup_{y'\in X} T(R(y,y'), B(y'))) \\ &= A \circ R \circ B \\ &= C(A,B) \end{aligned}$$

Conversely, we find, using the reflexivity of R

$$\begin{aligned} C(A,B) &= \sup_{y\in X} T(A(y), \sup_{y'\in X} T(R(y,y'), B(y'))) \\ &= \sup_{y\in X} T(A(y), \sup_{y'\in X} T(T(R(y,y'), R(y',y')), B(y'))) \\ &\leq \sup_{y\in X} T(A(y), \sup_{y'\in X} T(\sup_{x\in X} T(R(y,x), R(x,y')), B(y'))) \\ &= O(A,B) \end{aligned}$$

□

This again shows that fuzzy T-equivalence relations are not appropriate to model closeness in this context.

6 Concluding remarks

In practical applications, the semantics of the original RCC relations — and therefore of our generalized RCC relations — corresponding to a particular interpretation of C, are sometimes difficult to grasp. To cope with this, we

provided a characterization of the generalized RCC relations for the particular case where C is defined in terms of closeness between fuzzy sets in a suitable universe. This characterization paves the way for many applications, and shows that our framework is capable of tackling many of the limitations of the original RCC. Properties of the generalized RCC relations, such as the transitivity rules shown in [16], carry over to the specific interpretation discussed in this paper, yielding a sound (but incomplete) algorithm for spatial reasoning.

For the original RCC, alternative encodings using modal logic [5,7] and topological interpretations [9] have been used to obtain a better understanding of the meaning of the RCC relations. Apart from increasing the applicability of the RCC, such alternative encodings, and topological interpretations, have also led to important theoretical results about the RCC. This was possible because of the identification of particular canonical models of the RCC [9], i.e., structures in which interpretations of a set of regions $a, b, c, \ldots$, satisfying a consistent set of constraints like $NTPP(a, b) \vee PO(a, b)$, can always be found. An interesting question which we will study in future work is whether the interpretation we provided in this paper yields a canonical model of our fuzzy spatial relations. In other words, given a consistent set of constraints like $P(a, b) \geq 0.6 \vee O(b, c) \leq 0.3$, do there exist normalized fuzzy sets A, B, C, $\ldots$, (in a suitable universe X) for each of the variables a, b, c, $\ldots$ in the set of constraints, such that all constraints are satisfied?

A Proof of the characterization of the generalized RCC relations

In this appendix, we give a proof of the characterizations of the fuzzy relations P, O, and NTP that correspond to the definition of C given in (26). Recall that R is a resemblance relation, and T a left-continuous t-norm. First, we show a number of lemmas, related to the direct and superdirect image.

Lemma 3. *[10]*

$$R{\uparrow}{\downarrow}{\uparrow}A = R{\uparrow}A \tag{A.1}$$
$$R{\downarrow}{\uparrow}{\downarrow}A = R{\downarrow}A \tag{A.2}$$

Lemma 4. *[10] For any x in X, it holds that*

$$(R{\uparrow}{\downarrow}A)(x) \leq A(x) \leq (R{\downarrow}{\uparrow}A)(x) \tag{A.3}$$

Lemma 5. *Let E be defined as in Lemma 1, and let A be a fuzzy set in X. It holds that*

$$E{\uparrow}(R{\uparrow}A) = E{\downarrow}(R{\uparrow}A) = R{\uparrow}A$$
$$E{\uparrow}(R{\downarrow}A) = E{\downarrow}(R{\downarrow}A) = R{\downarrow}A$$

Proof. As an example, we show that $E{\downarrow}(R{\uparrow}A) = R{\uparrow}A$. We obtain, due to the reflexivity of E and (13),

$$\begin{aligned}(E{\downarrow}(R{\uparrow}A))(x) &= \inf_{y\in X} I_T(E(y,x), \sup_{z\in X} T(R(z,y), A(z))) \\ &\leq I_T(E(x,x), \sup_{z\in X} T(R(z,x), A(z))) \\ &= \sup_{z\in X} T(R(z,x), A(z)) \\ &= (R{\uparrow}A)(x)\end{aligned}$$

Conversely, using (12), (4), the symmetry of E and R, and (20), we find for x in X

$$\begin{aligned}(E{\downarrow}(R{\uparrow}A))(x) &= \inf_{y\in X} I_T(E(y,x), \sup_{z\in X} T(R(z,y), A(z))) \\ &\geq \inf_{y\in X} \sup_{z\in X} I_T(E(y,x), T(R(z,y), A(z))) \\ &\geq \inf_{y\in X} \sup_{z\in X} T(I_T(E(y,x), R(z,y)), A(z)) \\ &\geq \inf_{y\in X} \sup_{z\in X} T(R(z,x), A(z)) \\ &= \sup_{z\in X} T(R(z,x), A(z)) \\ &= (R{\uparrow}A)(x)\end{aligned}$$

□

Lemma 6. *Let A and B be fuzzy sets in X. It holds that*

$$incl(R{\uparrow}A, B) = incl(A, R{\downarrow}B) \tag{A.4}$$

$$incl(R{\uparrow}A, R{\uparrow}B) = incl(R{\downarrow}{\uparrow}A, R{\downarrow}{\uparrow}B) \tag{A.5}$$

Proof. First, note that (A.5) follows immediately from (A.1) and (A.4). Therefore, we only need to show (A.4):

$$\begin{aligned}&incl(R{\uparrow}A, B) \\ &= \inf_{x\in X} I_T(\sup_{y\in X} T(R(y,x), A(y)), B(x))\end{aligned}$$

By (9) and (5), we obtain

$$\begin{aligned}&= \inf_{x\in X} \inf_{y\in X} I_T(T(R(y,x), A(y)), B(x)) \\ &= \inf_{x\in X} \inf_{y\in X} I_T(A(y), I_T(R(y,x), B(x)))\end{aligned}$$

and finally by (10)

$$
\begin{aligned}
&= \inf_{y\in X} I_T(A(y), \inf_{x\in X} I_T(R(y,x), B(x))) \\
&= \inf_{y\in X} I_T(A(y), (R{\downarrow}B)(y)) \\
&= incl(A, R{\downarrow}B)
\end{aligned}
$$

□

Proof of Proposition 1

To prove (27), we first show that for an arbitrary region Z, it holds that

$$I_T(C(Z,A), C(Z,B)) \geq incl(R{\downarrow}{\uparrow}A, R{\downarrow}{\uparrow}B)$$

$$
\begin{aligned}
&I_T(C(Z,A), C(Z,B)) \\
&= I_T(Z \circ R \circ A, Z \circ R \circ B) \\
&= I_T(\sup_{x\in X} T(Z(x), \sup_{y\in X} T(R(x,y), A(y))), \sup_{x\in X} T(Z(x), \sup_{y\in X} T(R(x,y), B(y))))
\end{aligned}
$$

and by (9)

$$
\begin{aligned}
&= \inf_{x\in X} I_T(T(Z(x), \sup_{y\in X} T(R(x,y), A(y))), \sup_{x'\in X} T(Z(x'), \sup_{y\in X} T(R(x',y), B(y)))) \\
&\geq \inf_{x\in X} I_T(T(Z(x), \sup_{y\in X} T(R(x,y), A(y))), T(Z(x), \sup_{y\in X} T(R(x,y), B(y))))
\end{aligned}
$$

Finally, by (7), (3), and (A.5) we obtain

$$
\begin{aligned}
&\geq \inf_{x\in X} T(I_T(Z(x), Z(x)), I_T(\sup_{y\in X} T(R(x,y), A(y)), \sup_{y\in X} T(R(x,y), B(y)))) \\
&= \inf_{x\in X} I_T(\sup_{y\in X} T(R(x,y), A(y)), \sup_{y\in X} T(R(x,y), B(y))) \\
&= incl(R{\uparrow}A, R{\uparrow}B) \\
&= incl(R{\downarrow}{\uparrow}A, R{\downarrow}{\uparrow}B)
\end{aligned}
$$

By the definition of infimum as the greatest lower bound, we conclude that

$$P(A,B) = \inf_{Z\in U} I_T(C(Z,A), C(Z,B)) \geq incl(R{\downarrow}{\uparrow}A, R{\downarrow}{\uparrow}B) \tag{A.6}$$

Conversely, we find

$$
\begin{aligned}
P(A,B) &= \inf_{Z\in U} I_T(C(Z,A), C(Z,B)) \\
&= \inf_{Z\in U} I_T(\sup_{x\in X} T(Z(x), \sup_{y\in X} T(R(x,y), A(y))), \\
&\qquad \sup_{x\in X} T(Z(x), \sup_{y\in X} T(R(x,y), B(y))))
\end{aligned}
$$

For every z in X, we define the normalized fuzzy set S_z for x in X as

$$S_z(x) = \begin{cases} 1 & \text{if } x = z \\ 0 & \text{otherwise} \end{cases}$$

In other words, S_z corresponds to the crisp singleton set $\{z\}$. By monotonicity of the infimum, we find

$$\begin{aligned}
&\leq \inf_{z\in X} I_T(\sup_{x\in X} T(S_z(x), \sup_{y\in X} T(R(x,y), A(y))), \\
&\qquad \sup_{x\in X} T(S_z(x), \sup_{y\in X} T(R(x,y), B(y)))) \\
&= \inf_{z\in X} I_T(\sup_{y\in X} T(R(z,y), A(y)), \sup_{y\in X} T(R(z,y), B(y))) \\
&= incl(R{\uparrow}A, R{\uparrow}B)
\end{aligned}$$

Applying (A.5) to this last expression completes the proof of (27).

To prove (28), we first show that for an arbitrary region Z, it holds that

$$T(P(Z,A), P(Z,B)) \leq overl(R{\downarrow}{\uparrow}A, R{\downarrow}{\uparrow}B)$$

As we have defined regions as normalized fuzzy sets, there must exist an m in X for which $Z(m) = 1$. We obtain by (27) and (A.5)

$$\begin{aligned}
&T(P(Z,A), P(Z,B)) \\
&\quad = T(incl(R{\downarrow}{\uparrow}Z, R{\downarrow}{\uparrow}A), incl(R{\downarrow}{\uparrow}Z, R{\downarrow}{\uparrow}B)) \\
&\quad = T(incl(R{\uparrow}Z, R{\uparrow}A), incl(R{\uparrow}Z, R{\uparrow}B)) \\
&\quad = T(\inf_{x\in X} I_T(\sup_{y\in X} T(R(x,y), Z(y)), (R{\uparrow}A)(x)), \\
&\qquad \inf_{x\in X} I_T(\sup_{y\in X} T(R(x,y), Z(y)), (R{\uparrow}B)(x))) \\
&\quad \leq T(\inf_{x\in X} I_T(T(R(x,m), Z(m)), (R{\uparrow}A)(x)), \\
&\qquad \inf_{x\in X} I_T(T(R(x,m), Z(m)), (R{\uparrow}B)(x))) \\
&\quad = T(\inf_{x\in X} I_T(R(x,m), (R{\uparrow}A)(x)), \inf_{x\in X} I_T(R(x,m), (R{\uparrow}B)(x))) \\
&\quad \leq \sup_{y\in X} T(\inf_{x\in X} I_T(R(x,y), (R{\uparrow}A)(x)), \inf_{x\in X} I_T(R(x,y), (R{\uparrow}B)(x))) \\
&\quad = overl(R{\downarrow}{\uparrow}A, R{\downarrow}{\uparrow}B)
\end{aligned}$$

By the definition of the supremum as least upper bound, we conclude from this

$$O(A,B) = \sup_{z\in U} T(P(Z,A), P(Z,B)) \leq overl(R{\downarrow}{\uparrow}A, R{\downarrow}{\uparrow}B)$$

Conversely, we find by (27)

$$\begin{aligned}
O(A,B) &= \sup_{Z\in U} T(P(Z,A),P(Z,B)) \\
&= \sup_{Z\in U} T(\inf_{x\in X} I_T(\sup_{y\in X} T(R(x,y),Z(y)),(R{\uparrow}A)(x)), \\
&\qquad \inf_{x\in X} I_T(\sup_{y\in X} T(R(x,y),Z(y)),(R{\uparrow}B)(x))) \\
&\geq \sup_{z\in X} T(\inf_{x\in X} I_T(\sup_{y\in X} T(R(x,y),S_z(y)),(R{\uparrow}A)(x)), \\
&\qquad \inf_{x\in X} I_T(\sup_{y\in X} T(R(x,y),S_z(y)),(R{\uparrow}B)(x))) \\
&= \sup_{z\in X} T(\inf_{x\in X} I_T(R(x,z),(R{\uparrow}A)(x)), \inf_{x\in X} I_T(R(x,z),(R{\uparrow}B)(x))) \\
&= overl(R{\downarrow}{\uparrow}A, R{\downarrow}{\uparrow}B)
\end{aligned}$$

where the fuzzy set S_z is defined as before. This proves (28).

Finally, we prove (29). Let Z be an arbitrary region. We obtain by (28)

$$\begin{aligned}
&I_T(C(Z,A),O(Z,B)) \\
&\quad = I_T(\sup_{x\in X} T(Z(x),\sup_{y\in X} T(A(y),R(x,y))),\sup_{x\in X} T((R{\downarrow}{\uparrow}Z)(x),(R{\downarrow}{\uparrow}B)(x)))
\end{aligned}$$

By (9), we find

$$\begin{aligned}
&= \inf_{x\in X} I_T(T(Z(x),\sup_{y\in X} T(A(y),R(x,y))),\sup_{x'\in X} T((R{\downarrow}{\uparrow}Z)(x'),(R{\downarrow}{\uparrow}B)(x'))) \\
&\geq \inf_{x\in X} I_T(T(Z(x),\sup_{y\in X} T(A(y),R(x,y))),T((R{\downarrow}{\uparrow}Z)(x),(R{\downarrow}{\uparrow}B)(x)))
\end{aligned}$$

and by Lemma 4 and (5)

$$\begin{aligned}
&\geq \inf_{x\in X} I_T(T((R{\downarrow}{\uparrow}Z)(x),\sup_{y\in X} T(A(y),R(x,y))),T((R{\downarrow}{\uparrow}Z)(x),(R{\downarrow}{\uparrow}B)(x))) \\
&= \inf_{x\in X} I_T(\sup_{y\in X} T(A(y),R(x,y)),I_T((R{\downarrow}{\uparrow}Z)(x),T((R{\downarrow}{\uparrow}Z)(x),(R{\downarrow}{\uparrow}B)(x))))
\end{aligned}$$

Finally, using (4) and (3), we find

$$\begin{aligned}
&\geq \inf_{x\in X} I_T(\sup_{y\in X} T(A(y),R(x,y)),T(I_T((R{\downarrow}{\uparrow}Z)(x),(R{\downarrow}{\uparrow}Z)(x)),(R{\downarrow}{\uparrow}B)(x))) \\
&= \inf_{x\in X} I_T(\sup_{y\in X} T(A(y),R(x,y)),(R{\downarrow}{\uparrow}B)(x)) \\
&= incl(R{\uparrow}A, R{\downarrow}{\uparrow}B)
\end{aligned}$$

From the definition of infimum as the greatest lower bound, we conclude from this

$$NTP(A,B) = \inf_{Z\in U} I_T(C(Z,A),O(Z,B)) \geq incl(R{\uparrow}A, R{\downarrow}{\uparrow}B)$$

Conversely, we find by (28)

$$
\begin{aligned}
&NTP(A,B)\\
&= \inf_{Z\in U} I_T(C(Z,A), O(Z,B))\\
&= \inf_{Z\in U} I_T(C(Z,A), \sup_{x\in X} T((R{\downarrow}{\uparrow}Z)(x), (R{\downarrow}{\uparrow}B)(x)))\\
&\leq \inf_{z\in X} I_T(C(S_z,A), \sup_{x\in X} T((R{\downarrow}{\uparrow}S_z)(x), (R{\downarrow}{\uparrow}B)(x)))\\
&= \inf_{z\in X} I_T(C(S_z,A), \sup_{x\in X} T(\inf_{y\in X} I_T(R(x,y), \sup_{v\in X} T(R(y,v), S_z(v))), (R{\downarrow}{\uparrow}B)(x)))\\
&= \inf_{z\in X} I_T(C(S_z,A), \sup_{x\in X} T(\inf_{y\in X} I_T(R(x,y), R(y,z)), (R{\downarrow}{\uparrow}B)(x)))
\end{aligned}
$$

and by Lemma 1, Lemma 5, and the symmetry of C

$$
\begin{aligned}
&= \inf_{z\in X} I_T(C(S_z,A), \sup_{x\in X} T(E(x,z), (R{\downarrow}{\uparrow}B)(x)))\\
&= \inf_{z\in X} I_T(C(A,S_z), (R{\downarrow}{\uparrow}B)(z))\\
&= \inf_{z\in X} I_T(\sup_{x\in X} T(A(x), \sup_{y\in X} T(R(x,y), S_z(y))), (R{\downarrow}{\uparrow}B)(z))\\
&= \inf_{z\in X} I_T(\sup_{x\in X} T(A(x), R(x,z)), (R{\downarrow}{\uparrow}B)(z))\\
&= \inf_{z\in X} I_T((R{\uparrow}A)(z), (R{\downarrow}{\uparrow}B)(z))\\
&= incl(R{\uparrow}A, R{\downarrow}{\uparrow}B)
\end{aligned}
$$

which concludes the proof of (29).

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