Intuitionistic Fuzzy Modal Logic Operations: a New Outlook

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Abstract: New properties of intuitionistic fuzzy modal logic operators are discussed and these operators are interpreted as modifications of triangular norms and conorms, and as particular kinds of aggregation operations.

1 Introduction

Intuitionistic Fuzzy Sets (IFSs) were introduced in 1983 in [1] as extensions of ordinary fuzzy sets. Short remarks on Intuitionistic Fuzzy Logics (IFLs) are discussed in e.g. [2, 3, 4], while algebraic advances in the construction and classification of logical connectives for use in IFLs can be found in e.g. [5, 6, 7, 8]. The most important part of the IFSs and IFLs theory is related to the modal operators, which are extensions of the classical modal logic operators $\Box$ and $\Diamond$. Here, these IFL operators are interpreted as modifications of triangular norms and conorms, some of their basic properties are discussed, and their use as aggregation operators is shown. A bibliography of the publications over IFSs and IFLs is published in [12].

2 Short remarks on IFL

Following [2, 3, 4] we give some elements of IFL. Two real numbers, $\mu(p)$ and $\nu(p)$, are assigned to the proposition p with the following constraint to hold:

$$\mu(p) + \nu(p) \leq 1.$$

They correspond to the "truth degree" and to the "falsity degree" of p. Let this assignment be provided by an evaluation function V, defined over a set of propositions S in such a way that:

$$V(p) = \langle \mu(p),\ \nu(p) \rangle.$$

When values $V(p)$ and $V(q)$ of the propositions p and q are known, the evaluation function V can be also extended for the operations "negation" ($\neg$), "conjunction" (&), "disjunction" ($\vee$), “implication” ($\supset$) and others, e.g., through the definitions:

$$
\begin{aligned}
V(\neg p) &= \langle \nu(p), \mu(p) \rangle, \\
V(p \,\&\, q) &= \langle \min(\mu(p), \mu(q)), \max(\nu(p), \nu(q)) \rangle, \\
V(p \vee q) &= \langle \max(\mu(p), \mu(q)), \min(\nu(p), \nu(q)) \rangle, \\
V(p \supset q) &= \langle \max(\nu(p), \mu(q)), \min(\mu(p), \nu(q)) \rangle.
\end{aligned} \tag{1}
$$

Let for every proposition p: $V(p) = \langle \mu(p), \nu(p) \rangle$. Then p is called a *(standard) tautology* if and only if $\mu(p) = 1$ and $\nu(p) = 0$. We say that p is an *intuitionistic fuzzy tautology (IFT)* if and only if $\mu(p) \geq \nu(p)$. Next, consider the following partially ordered set $(L^*, \leq_{L^*})$:

$$
\begin{gathered}
L^* = \{\langle x_1, x_2 \rangle \mid (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\}, \\
\langle x_1, x_2 \rangle \leq_{L^*} \langle y_1, y_2 \rangle \text{ if and only if } x_1 \leq y_1 \text{ and } x_2 \geq y_2.
\end{gathered}
$$

Then $(L^*, \leq_{L^*})$ is a complete lattice [9]. We denote its units as $0_{L^*} = \langle 0, 1 \rangle$ and $1_{L^*} = \langle 1, 0 \rangle$. Clearly, the evaluation function V can be seen as an $S \to L^*$ mapping, where S is a given set of propositions. The geometrical interpretation of a proposition p and $V(p) = \langle \mu(p), \nu(p) \rangle$ are shown in Fig. 1; the triangle represents the set L^*. Define, for all $x = \langle x_1, x_2 \rangle \in L^*$ and $y = \langle y_1, y_2 \rangle \in L^*$:

$$
\begin{aligned}
\neg x &= \langle x_2, x_1 \rangle, \\
x \,\&\, y &= \langle \min(x_1, y_1), \max(x_2, y_2) \rangle, \\
x \vee y &= \langle \max(x_1, y_1), \min(x_2, y_2) \rangle, \\
x \supset y &= \langle \max(x_2, y_1), \min(x_1, y_2) \rangle.
\end{aligned}
$$

Then it is easy to see that, for any two propositions $p, q \in S$: $V(\neg p) = \neg(V(p))$, $V(p \,\&\, q) = V(p) \,\&\, V(q)$, $V(p \vee q) = V(p) \vee V(q)$ and $V(p \supset q) = V(p) \supset V(q)$.

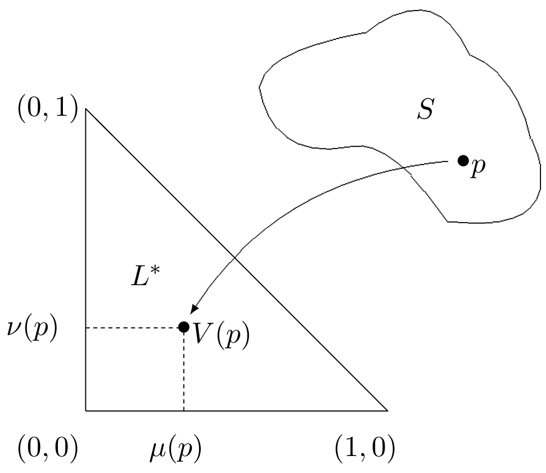

Fig. 1

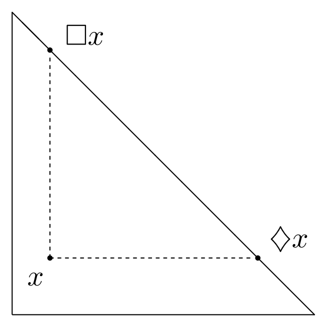

Fig. 2

Define the modal operators “$\Box$” and “$\Diamond$” as follows, for $x = \langle x_1, x_2 \rangle \in L^*$:

$$
\begin{aligned}
\Box(x) &= \langle x_1, 1 - x_1 \rangle, \\
\Diamond(x) &= \langle 1 - x_2, x_2 \rangle.
\end{aligned}
$$

Their geometrical interpretations are shown in Fig. 2. The evaluation function V can be extended also for the modal operators as follows

$$V(\Box p) = \Box(V(p)) = \langle \mu(p), 1 - \mu(p) \rangle,$$
$$V(\Diamond p) = \Diamond(V(p)) = \langle 1 - \nu(p), \nu(p) \rangle.$$

It can be seen easily that for each proposition p such that $V(p) = \langle \mu(p), 1 - \mu(p) \rangle$, i.e., the estimation is fuzzy, but not purely intuitionistic fuzzy, then $V(\Diamond p) = V(p) = V(\Box p)$.

3 On the extended negation, conjunction, disjunction and implication

In [6, 7, 8] the following operations on L^* were defined (a more restricted version was given earlier in [5]).

Definition 1: *A triangular norm (t-norm) on L^* is a commutative, associative, increasing mapping $\mathcal{T} : (L^*)^2 \to L^*$ such that $\mathcal{T}(1_{L^*}, x) = x$, for all $x \in L^*$.*

A triangular conorm (t-conorm) on L^* is a commutative, associative, increasing mapping $\mathcal{S} : (L^*)^2 \to L^*$ such that $\mathcal{S}(0_{L^*}, x) = x$, for all $x \in L^*$.

A negator on L^* is a decreasing mapping $\mathcal{N} : L^* \to L^*$ such that $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$.

An implicator on L^* is a mapping $\mathcal{I} : (L^*)^2 \to L^*$ such that $\mathcal{I}(0_{L^*}, 0_{L^*}) = \mathcal{I}(0_{L^*}, 1_{L^*}) = \mathcal{I}(1_{L^*}, 1_{L^*}) = 1_{L^*}$, $\mathcal{I}(1_{L^*}, 0_{L^*}) = 0_{L^*}$, $\mathcal{I}(., x)$ is decreasing and $\mathcal{I}(x, .)$ is increasing, for all $x \in L^*$.

Using these operators it is possible to extend the evaluation of "negation", "conjunction", "disjunction" and "implication" as follows:

$$V(\neg p) = \mathcal{N}(V(p)) = \mathcal{N}(\langle \mu(p), \nu(p) \rangle),$$
$$V(p \,\&\, q) = \mathcal{T}(V(p), V(q)) = \mathcal{T}(\langle \mu(p), \nu(p) \rangle, \langle \mu(q), \nu(q) \rangle),$$
$$V(p \vee q) = \mathcal{S}(V(p), V(q)) = \mathcal{S}(\langle \mu(p), \nu(p) \rangle, \langle \mu(q), \nu(q) \rangle),$$
$$V(p \supset q) = \mathcal{I}(V(p), V(q)) = \mathcal{I}(\langle \mu(p), \nu(p) \rangle, \langle \mu(q), \nu(q) \rangle),$$

4 On the extended modal IFL operators

Let $\alpha, \beta \in [0, 1]$. Following [2, 4, 3], we define the operators D_α, $F_{\alpha,\beta}$ (for $\langle \alpha, \beta \rangle \in L^*$), $G_{\alpha,\beta}$, $H_{\alpha,\beta}$, $H^*_{\alpha,\beta}$, $J_{\alpha,\beta}$ and $J^*_{\alpha,\beta}$ on L^* by, for all $x = \langle x_1, x_2 \rangle \in L^*$:

$$D_\alpha(x) = \langle x_1 + \alpha.(1 - x_1 - x_2), x_2 + (1 - \alpha).(1 - x_1 - x_2) \rangle,$$
$$F_{\alpha,\beta}(x) = \langle x_1 + \alpha.(1 - x_1 - x_2), x_2 + \beta.(1 - x_1 - x_2) \rangle,$$
$$G_{\alpha,\beta}(x) = \langle \alpha.x_1, \beta.x_2 \rangle,$$
$$H_{\alpha,\beta}(x) = \langle \alpha.x_1, x_2 + \beta.(1 - x_1 - x_2) \rangle,$$
$$H^*_{\alpha,\beta}(x) = \langle \alpha.x_1, x_2 + \beta.(1 - \alpha.x_1 - x_2) \rangle,$$
$$J_{\alpha,\beta}(x) = \langle x_1 + \alpha.(1 - x_1 - x_2), \beta.x_2 \rangle,$$
$$J^*_{\alpha,\beta}(x) = \langle x_1 + \alpha.(1 - x_1 - \beta.x_2), \beta.x_2 \rangle.$$

For any $O \in \{D_\alpha, F_{\alpha,\beta}, G_{\alpha,\beta}, H_{\alpha,\beta}, H^*_{\alpha,\beta}, J_{\alpha,\beta}, J^*_{\alpha,\beta}\}$ and any proposition p, we define $O(p)$ by

$$V(O(p)) = O(V(p)) = O(\langle \mu(p), \nu(p) \rangle),$$

e.g. for $O = D_\alpha$ we obtain $V(D_\alpha(p)) = \langle \mu(p) + \alpha.(1 - \mu(p) - \nu(p)), \nu(p) + (1-\alpha).(1 - \mu(p) - \nu(p)) \rangle$.

The geometrical interpretations of the seven operators are given in Fig. 3–9, where in Fig. 6–9 the shaded area's indicate the possible positions of the quantities.

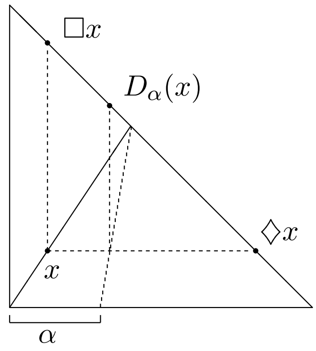

Fig. 3

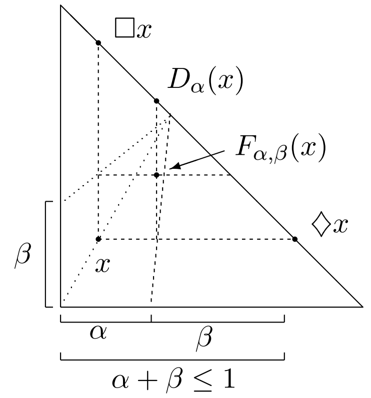

Fig. 4

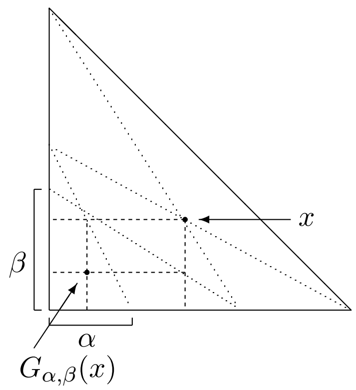

Fig. 5

Obviously, for all $x \in L^*$ and $\alpha \in [0,1]$,

$$\Box x = D_0(x),$$
$$\Diamond x = D_1(x),$$
$$D_\alpha(x) = F_{\alpha,1-\alpha}(x).$$

Theorem 1: *Let $\alpha, \beta \in [0,1]$ and let the operations &, $\vee$ and $\supset$ be as in (1). Then for every two propositions p and q it holds that*

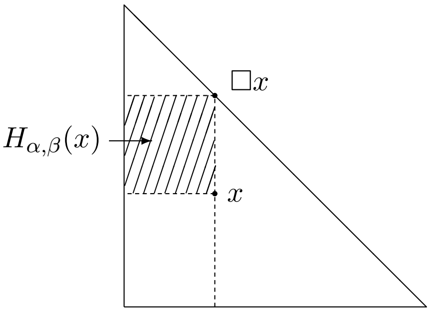

Fig. 6

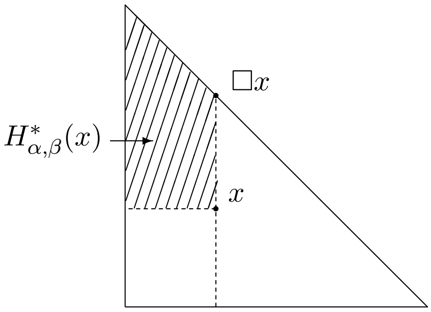

Fig. 7

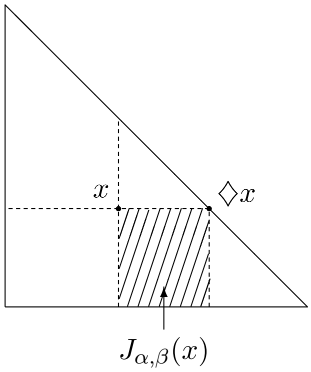

Fig. 8

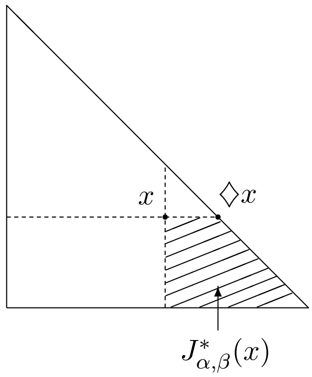

Fig. 9

(a) $F_{\alpha,\beta}(p \,\&\, q) \supset F_{\alpha,\beta}(p) \,\&\, F_{\alpha,\beta}(q)$, *for* $\langle \alpha, \beta \rangle \in L^*$,

(b) $F_{\alpha,\beta}(p \vee q) \subset F_{\alpha,\beta}(p) \vee F_{\alpha,\beta}(q)$, *for* $\langle \alpha, \beta \rangle \in L^*$,

(c) $G_{\alpha,\beta}(p \,\&\, q) \Leftrightarrow G_{\alpha,\beta}(p) \,\&\, G_{\alpha,\beta}(q)$,

(d) $G_{\alpha,\beta}(p \vee q) \Leftrightarrow G_{\alpha,\beta}(p) \vee G_{\alpha,\beta}(q)$,

(e) $H_{\alpha,\beta}(p \,\&\, q) \supset H_{\alpha,\beta}(p) \,\&\, H_{\alpha,\beta}(q)$,

(f) $H_{\alpha,\beta}(p \vee q) \subset H_{\alpha,\beta}(p) \vee H_{\alpha,\beta}(q)$,

(g) $J_{\alpha,\beta}(p \,\&\, q) \subset H_{\alpha,\beta}(p) \,\&\, H_{\alpha,\beta}(q)$,

(h) $H_{\alpha,\beta}(p \vee q) \supset H_{\alpha,\beta}(p) \vee H_{\alpha,\beta}(q)$,

(i) $H^*_{\alpha,\beta}(p \,\&\, q) \supset H^*_{\alpha,\beta}(p) \,\&\, H^*_{\alpha,\beta}(q)$,

(j) $H^*_{\alpha,\beta}(p \vee q) \subset H^*_{\alpha,\beta}(p) \vee H^*_{\alpha,\beta}(q)$,

(k) $J^*_{\alpha,\beta}(p \,\&\, q) \subset H^*_{\alpha,\beta}(p) \,\&\, H^*_{\alpha,\beta}(q)$,

(l) $H^*_{\alpha,\beta}(p \vee q) \supset H^*_{\alpha,\beta}(p) \vee H^*_{\alpha,\beta}(q)$

are IFTs (here we use the notation $p \subset q$ *if and only if* $q \supset p$*, and* $p \Leftrightarrow q$ *if and only if* $p \supset q$ *and* $q \supset p$*).*

Proof. As an example we prove (a). Let $V(p) = \langle a, b\rangle \in L^*$ and $V(q) = \langle c, d\rangle \in L^*$. Then

$$\begin{aligned}
&V(F_{\alpha,\beta}(p \,\&\, q) \supset F_{\alpha,\beta}(p) \,\&\, F_{\alpha,\beta}(q))\\
&= \langle \min(a, c) + \alpha.(1 - \min(a, c) - \max(b, d)), \max(b, d)+\\
&\quad \beta.(1 - \min(a, c) - \max(b, d))\rangle \supset \langle \min(a + \alpha.(1 - a - b), c+\\
&\quad \alpha.(1 - c - d)), \max(b + \beta.(1 - a - b), d + \beta.(1 - c - d))\rangle\\
&= \langle \max(\max(b, d) + \beta.(1 - \min(a, c) - \max(b, d)), \min(a+\\
&\quad \alpha.(1 - a - b), c + \alpha.(1 - c - d))), \min(\min(a, c) + \alpha.(1 - \min(a, c)\\
&\quad - \max(b, d)), \max(b + \beta.(1 - a - b), d + \beta.(1 - c - d)))\rangle.
\end{aligned}$$

and

$$\begin{aligned}
&\max(\max(b, d) + \beta.(1 - \min(a, c) - \max(b, d)), \min(a+\\
&\alpha.(1 - a - b), c + \alpha.(1 - c - d))) - \min(\min(a, c) + \alpha.(1-\\
&\min(a, c) - \max(b, d)), \max(b + \beta.(1 - a - b), d + \beta.(1 - c - d)))\\
&\geq \max(b, d) + \beta.(1 - \min(a, c) - \max(b, d))\\
&\quad - \max(b + \beta.(1 - a - b), d + \beta.(1 - c - d)) \geq 0,
\end{aligned}$$

i.e. $F_{\alpha,\beta}(p \,\&\, q) \supset F_{\alpha,\beta}(p) \,\&\, F_{\alpha,\beta}(q)$ is an IFT. □

Theorem 2: *For any $x \in L^*$ and for every $\alpha, \beta, \alpha', \beta' \in [0, 1]$ such that $\alpha \leq \alpha'$ and $\beta \geq \beta'$, it holds:*

(a) $D_\alpha(x) \leq_{L^*} D_{\alpha'}(x)$,

(b) $F_{\alpha,\beta}(x) \leq_{L^*} F_{\alpha',\beta'}(x)$, *for* $\langle\alpha, \beta\rangle \in L^*$ *and* $\langle\alpha', \beta'\rangle \in L^*$,

(c) $G_{\alpha,\beta}(x) \leq_{L^*} G_{\alpha',\beta'}(x)$,

(d) $H_{\alpha,\beta}(x) \leq_{L^*} H_{\alpha',\beta'}(x)$,

(e) $H^*_{\alpha,\beta}(x) \leq_{L^*} H^*_{\alpha',\beta'}(x)$,

(f) $J_{\alpha,\beta}(x) \leq_{L^*} J_{\alpha',\beta'}(x)$,

(g) $J^*_{\alpha,\beta}(x) \leq_{L^*} J^*_{\alpha',\beta'}(x)$.

Theorem 3: *For any $\alpha, \beta \in [0, 1]$, D_α, $F_{\alpha,\beta}$ (for $\langle\alpha, \beta\rangle \in L^*$), $G_{\alpha,\beta}$, $H_{\alpha,\beta}$, $H^*_{\alpha,\beta}$, $J_{\alpha,\beta}$, $J^*_{\alpha,\beta}$ are increasing $L^* \to L^*$ mappings.*

Theorem 4: *For any $x \in L^*$ and for every $\alpha, \beta \in [0, 1]$, it holds:*

(a) $\neg D_\alpha(x) = D_{1-\alpha}(\neg x)$,

(b) $\neg F_{\alpha,\beta}(x) = F_{\beta,\alpha}(\neg x)$, *for* $\langle\alpha, \beta\rangle \in L^*$,

(c) $\neg G_{\alpha,\beta}(x) = G_{\beta,\alpha}(\neg x)$,

(d) $\neg H_{\alpha,\beta}(x) = J_{\beta,\alpha}(\neg x)$,

(e) $\neg J_{\alpha,\beta}(x) = H_{\beta,\alpha}(\neg x)$,

(f) $\neg H^*_{\alpha,\beta}(x) = J^*_{\beta,\alpha}(\neg x)$,

(g) $\neg J^*_{\alpha,\beta}(x) = H^*_{\beta,\alpha}(\neg x)$.

5 Triangular norms, conorms and extended modal IFL operators

First note that the following assertions can be proved as above.

Theorem 5: *For any $x \in L^*$, it holds that*

$$F_{0,0}(x) = G_{1,1}(x) = H_{1,0}(x) = H^*_{1,0}(x) = J_{0,1}(x) = J^*_{0,1}(x) = x.$$

Theorem 6: *For any $\langle a,b\rangle, \langle c,d\rangle \in L^*$, it holds that*

$$G_{a,b}(\langle c,d\rangle) = G_{c,d}(\langle a,b\rangle).$$

Theorem 7: *For any $\langle a,b\rangle, \langle c,d\rangle \in L^*$ it holds that*

(a) $F_{a,b}(\langle c,d\rangle) = F_{c,d}(\langle a,b\rangle)$, *if* $ad = bc$,

(b) $H_{a,b}(\langle c,d\rangle) = H_{c,d}(\langle a,b\rangle)$, *if* $ad = bc$,

(c) $H^*_{a,b}(\langle c,d\rangle) = H^*_{c,d}(\langle a,b\rangle)$, *if* $b = d$,

(d) $J_{a,b}(\langle c,d\rangle) = J_{c,d}(\langle a,b\rangle)$, *if* $ad = bc$,

(e) $J^*_{a,b}(\langle c,d\rangle) = J^*_{c,d}(\langle a,b\rangle)$, *if* $a = c$.

Theorem 8: *For any $\langle a,b\rangle, \langle c,d\rangle, \langle e,f\rangle \in L^*$, it holds that*

(a) $F_{F_{a,b}(\langle c,d\rangle)}(\langle e,f\rangle) = F_{a,b}(F_{c,d}(\langle e,f\rangle))$,

(b) $G_{G_{a,b}(\langle c,d\rangle)}(\langle e,f\rangle) = G_{a,b}(G_{c,d}(\langle e,f\rangle))$.

In general the following equalities are not satisfied:

(a) $H_{H_{a,b}(\langle c,d\rangle)}(\langle e,f\rangle) = H_{a,b}(H_{c,d}(\langle e,f\rangle))$,

(b) $H^*_{H^*_{a,b}(\langle c,d\rangle)}(\langle e,f\rangle) = H^*_{a,b}(H^*_{c,d}(\langle e,f\rangle))$,

(c) $J_{J_{a,b}(\langle c,d\rangle)}(\langle e,f\rangle) = J_{a,b}(J_{c,d}(\langle e,f\rangle))$,

(d) $J^*_{J^*_{a,b}(\langle c,d\rangle)}(\langle e,f\rangle) = J^*_{a,b}(J^*_{c,d}(\langle e,f\rangle))$.

Let, for any $\langle \alpha,\beta\rangle \in L^*$, $O_{\alpha,\beta} \in \{F_{\alpha,\beta}, G_{\alpha,\beta}, H_{\alpha,\beta}, H^*_{\alpha,\beta}, J_{\alpha,\beta}, J^*_{\alpha,\beta}\}$. Define now the $(L^*)^2 \to L^*$ mapping O as $O(\langle a,b\rangle, \langle c,d\rangle) = O_{a,b}(\langle c,d\rangle)$, for all $\langle a,b\rangle, \langle c,d\rangle \in L^*$. Theorem 6 and Theorem 7 give the conditions for the commutativity of O. In Theorem 8 the associativity of O is discussed. From Theorem 2 and Theorem 3 it follows that O is increasing. The boundary conditions are discussed in Theorem 5. From these theorems it is clear that the modal operators can be transformed into operators which satisfy similar properties as t-norms and t-conorms on L^*; it is important to note however that *none of them is actually either a t-norm or t-conorm on* L^*.

We will now show that the new operators belong to a more general class, namely the class of aggregation operators on L^*. Aggregation operators on [0,1] were studied e.g. in [10, 11]. A possible generalization to L^* is given by:

Definition 2: *An aggregation operator on L^* is a mapping $\mathcal{A} : (L^*)^2 \to L^*$ such that*

(A1) *for all $x, x', y, y' \in L^*$, if $x \leq_{L^*} x'$ and $y \leq_{L^*} y'$, then $\mathcal{A}(x,y) \leq_{L^*} \mathcal{A}(x',y')$,* *(A2)* $\mathcal{A}(0_{L^*}, 0_{L^*}) = 0_{L^*}$,

(A3) $\mathcal{A}(1_{L^*}, 1_{L^*}) = 1_{L^*}$.

Theorem 9: *Let $O \in \{F, G, H, H^*, J, J^*\}$, then $O(0_{L^*}, 0_{L^*}) = 0_{L^*}$ and $O(1_{L^*}, 1_{L^*}) = 1_{L^*}$.*

From Theorem 9, Theorem 2 and Theorem 3 it follows that the modal operators can be transformed into aggregation operators on L^*. From Theorem 4 it follows that these aggregation operators satisfy some duality properties, i.e. for all $x, y \in L^*$:

$$\begin{aligned} F(x,y) &= \neg F(\neg x, \neg y), \\ G(x,y) &= \neg G(\neg x, \neg y), \\ H(x,y) &= \neg J(\neg x, \neg y), \\ H^*(x,y) &= \neg J^*(\neg x, \neg y). \end{aligned}$$

6 Conclusion

In this paper we discussed the properties of the extended modal operators and starting from them we constructed new binary operators on L^* which have similar properties as t-norms and t-conorms. We also showed that these operators belong to the more general class of aggregation operators on L^*.

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