Chapter 3

THE GENERALIZED MODUS PONENS IN A FUZZY SET THEORETICAL FRAMEWORK

Chris Cornelis, Martine De Cock* and Etienne E. Kerre

Department of Applied Mathematics and Computer Science

Ghent University

Krijgslaan 281 (S9), B-9000 Ghent, Belgium

{Chris.Cornelis, Martine.DeCock, Etienne.Kerre}@rug.ac.be

Abstract Starting from an input fuzzy set and an IF–THEN rule, implementations of the Generalized Modus Ponens (GMP) in a fuzzy set theoretical framework allow the derivation of an output fuzzy set. If the GMP is implemented by means of the Compositional Rule of Inference (CRI) in general this derivation involves a lot of computational efforts, and the shape of the resulting membership function often seems quite arbitrary compared to that of the input membership function. In this chapter we present a review of techniques, generating as their output a fuzzy set belonging to a predefined class, that are not afflicted with these disadvantages.

Keywords: generalized modus ponens, compositional rule of inference, closed system, linguistic hedge

1. INTRODUCTION

The modus ponens (MP)

$$\begin{array}{l} \mathsf{x} \text{ is } A \\ \text{IF } \mathsf{x} \text{ is } A \text{ THEN } \mathsf{y} \text{ is } B \\ \hline \mathsf{y} \text{ is } B \end{array} \qquad \text{(SCHEME–1)}$$

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is a well–known deduction rule in (boolean) logic. From the fact "x is A" and the IF–THEN rule "IF x is A THEN y is B," we can derive a new fact, namely "y is B." However, if we do not exactly know that "x is A," we cannot make any deduction concerning y, even if we would have tons of other information on x. This implies that if we want to develop a useful derivation system (e.g., for a computer–controlled car) based on this MP, we have to provide an IF–THEN rule for each possible A. Needless to say this would be highly inefficient, if not impossible.

A better solution would be to use an alternative derivation rule, called the generalized modus ponens (GMP):

$$\begin{array}{l} \text{x is } A' \\ \text{IF x is } A \text{ THEN y is } B \\ \hline \text{y is } B' \end{array} \qquad \text{(SCHEME–2)}$$

In this scheme it is possible to make a deduction concerning y even if the fact we know about x (namely "x is A'") does not exactly match the premise of the rule (namely, "x is A"). B' will of course depend on A', A and B. We postpone the discussion of how to derive B' to the following sections.

If x is a variable taking its values in a universe X, and y a variable taking its values in a universe Y, then in a fuzzy set theoretical framework, A' and A are usually represented by fuzzy sets on X, while B and B' are modelled by fuzzy sets on Y.

Definition 1 (fuzzy set) *A fuzzy set A on a universe X is a $X-[0,1]$ mapping. This mapping is often referred to as the membership function of A. For all x in X, $A(x)$ is called the degree of membership of x in A. The class of all fuzzy sets on X is denoted $\mathcal{F}(X)$. A fuzzy relation from a universe X to a universe Y is a fuzzy set on $X \times Y$, i.e., an element of $\mathcal{F}(X \times Y)$. Furthermore for A and B in $\mathcal{F}(X)$:*

$$A \subseteq B \Leftrightarrow (\forall x \in X)(A(x) \leq B(x))$$

$$support(A) = \{x | x \in X \text{ and } 0 < A(x)\}$$

$$shell(A) = \{x | x \in X \text{ and } 0 < A(x) < 1\}$$

$$(\forall x \in X)((A \cup B)(x) = max(A(x), B(x)))$$

$$(\forall x \in X)((A \cap B)(x) = min(A(x), B(x)))$$

In applications A' corresponds either to a crisp object a of the universe (e.g.,a temperature, a speed,...) or to a linguistic term <term> (e.g., *"warm," "slow,"*...). In the first case $A' = \{a\}$, while in the second

case for each x in X, $A'(x)$ is the degree to which x satisfies <term>. Likewise A and B represent objects or terms (the latter being much more frequent). However, while A', A and B are *constructed* to model objects or terms, B' is a fuzzy set *derived* during the inference process (in a way we still have to explain). B' is the representation of "some" object or linguistic term, but there is often no obvious way in determining which one.

If the output of the inference process is meant to control a machine, it is preferable that B' is associated to a crisp object of the universe (most often a crisp number). This association process is called defuzzification (see e.g., [10]). However, if the output is meant to guide a person, B' would better be interpreted linguistically. To do this, there is often a need for some process of linguistic approximation (see e.g., [12]). In this chapter we will only focus on implementations of the GMP generating an output that can be interpreted as a linguistic term without further need for linguistic approximation.

2. THE COMPOSITIONAL RULE OF INFERENCE

The most popular way of deriving B' from A', A and B is no doubt the compositional rule of inference (CRI) introduced by Zadeh [12]. First we recall the notions of triangular norm and implicator.

Definition 2 (triangular norm) *A triangular norm $\mathcal{T}$ (or shortly t-norm) is an associative and commutative increasing*[1] $[0,1]^2 - [0,1]$ *mapping that satisfies the boundary conditions $\mathcal{T}(1,1) = 1$, $\mathcal{T}(1,0) = \mathcal{T}(0,1) = \mathcal{T}(0,0) = 0$ and $(\forall x \in [0,1])$ $(\mathcal{T}(1,x) = \mathcal{T}(x,1) = x)$.*

Definition 3 (implicator) *An implicator $\mathcal{I}$ is a hybrid monotonic* $[0,1]^2 - [0,1]$ *mapping (i.e., $(\forall x \in [0,1])$ $(\mathcal{I}(.,x)$ is decreasing and $\mathcal{I}(x,.)$ is increasing)) that satisfies the boundary conditions $\mathcal{I}(1,0) = 0$, $\mathcal{I}(1,1) = \mathcal{I}(0,1) = \mathcal{I}(0,0) = 1$.*

For a particular t–norm (e.g., Table 1.1) , we can associate with each fact "x and y have relation R" an $\mathcal{F}(X) - \mathcal{F}(Y)$ mapping.

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[1] We recall that for a poset $(X, \leq)$ and f a $X - [0,1]$ mapping :

$$f \text{ is increasing} \Leftrightarrow (\forall (x,y) \in X^2)(x \leq y \Rightarrow f(x) \leq f(y))$$

$$f \text{ is decreasing} \Leftrightarrow (\forall (x,y) \in X^2)(x \leq y \Rightarrow f(x) \geq f(y))$$

The minimum operator $\mathcal{T}_M$	$\mathcal{T}_M(x,y) = min(x,y)$
The algebraic product $\mathcal{T}_P$	$\mathcal{T}_P(x,y) = x.y$
The Lukasiewicz t-norm $\mathcal{T}_W$	$\mathcal{T}_W(x,y) = max(0, x+y-1)$

Table 1.1 Examples of popular t-norms ($(x,y) \in [0,1]^2$).

The Kleene–Dienes implicator $\mathcal{I}_{KD}$	$\mathcal{I}_{KD}(x,y) = max(1-x, y)$
The Lukasiewicz implicator $\mathcal{I}_W$	$\mathcal{I}_W(x,y) = min(1, 1-x+y)$
The Reichenbach implicator $\mathcal{I}_R$	$\mathcal{I}_R(x,y) = 1-x+xy$
The Standard Strict implicator $\mathcal{I}_s$	$\mathcal{I}_s(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & elsewhere \end{cases}$
The Standard Star implicator $\mathcal{I}_g$	$\mathcal{I}_g(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ y & elsewhere \end{cases}$
The Gaines implicator $\mathcal{I}_\Delta$	$\mathcal{I}_\Delta(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & elsewhere \end{cases}$

Table 1.2 Examples of popular implicators ($(x,y) \in [0,1]^2$).

Definition 4 (compositional rule of inference) *If $\mathcal{T}$ is a t-norm, $R \in \mathcal{F}(X \times Y)$, then the $\mathcal{F}(X) - \mathcal{F}(Y)$ mapping $cri_R^{\mathcal{T}}$ corresponding to the fact*

x and y have relation R

is defined by for $A' \in \mathcal{F}(X)$, $y \in Y$:

$$cri_R^{\mathcal{T}}(A')(y) = \sup_{x \in X} \mathcal{T}(A'(x), R(x,y))$$

The application of the operator $cri_R^{\mathcal{T}}$ to a fuzzy set A', generating a new fuzzy set, is called the compositional rule of inference.

The rule "IF x is A THEN y is B" states a relation between x and y. This relation can be expressed, by means of an implicator $\mathcal{I}$ (e.g., Table 1.2), by $A \Rightarrow_{\mathcal{I}} B$ which is defined as (for $x \in X$, $y \in Y$):

$$(A \Rightarrow_{\mathcal{I}} B)(x,y) = \mathcal{I}(A(x), B(y))$$

The B' in (SCHEME–2) can now be computed using the *cri*–operator defined above, namely $B' = cri^{\mathcal{T}}_{A \Rightarrow_{\mathcal{I}} B}(A')$.

3. THE IMPORTANCE OF BEING CLOSED

Much effort has already been put in investigating various techniques of inference based on the compositional rule of inference [4], and also on similarity relations [11], on fuzzy truth values [1], on interval-based implications [9], ... resulting in an extensive and detailed literature. In practice, however, most of these often sophisticated methods are being sparsely used because of the high computational demands they pose. The CRI requires that, for each evaluation of the output fuzzy set, the supremum over a possibly large set of values is computed. When the input is crisp (e.g., a crisp temperature) calculating the supremum can be avoided. However, when the input is fuzzy (e.g., a linguistic term), there is no straightforward way to eliminate the calculation of the supremum. As a consequence, most working systems involve simple computations transforming crisp inputs into crisp outputs, a task for which efficient algorithms have been devised. Intelligent systems interacting with humans, on the other hand, should be expected to cope with imprecise information conveyed by a human operator under the form of linguistic statements; to be meaningful, the answer produced by the system should preferably be stated in the user's own natural language (i.e., be of a linguistic nature).

The class of fuzzy sets commonly encountered in computer applications is often restricted to a subclass of fuzzy sets that can all be modelled by the same general shape membership function characterized by a small number of parameters. In that case all membership functions can be kept in memory by storing the parameters instead of all the membership values. If operations are performed on these fuzzy sets, it is desirable that the resulting fuzzy sets can also be characterized by this general shape function. This brings us to the concept of a closed system.

Definition 5 (closed system) *For $U \subseteq \mathcal{F}(X)$, $V \subseteq \mathcal{F}(Y)$ and m a $\mathcal{F}(X) - \mathcal{F}(Y)$ mapping (i.e., an operator transforming a fuzzy set on X into a fuzzy set on Y), (U,V,m) is called a closed system iff $(\forall A \in U)(m(A) \in V)$.*

Applied to inferencing, U and V constitute classes of fuzzy sets characterized by a (possibly but not necessarily different) general shape function, while m corresponds to an implementation of the IF–THEN rule (or more general: of the fact "x and y have relation R"). We begin by remarking that the implementation of the GMP by means of the CRI as described above gives rise to a closed system if all the fuzzy sets on X and on Y are taken into account.

Theorem 1 *For every t-norm $\mathcal{T}$, every implicator $\mathcal{I}$, for all $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$:*

$$(\mathcal{F}(X), \mathcal{F}(Y), cri^{\mathcal{T}}_{A \Rightarrow_{\mathcal{I}} B}) \text{ is a closed system.}$$

In this chapter, however, we will discuss three closed systems (U, V, m) in which U and V are relatively small subsets of $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ respectively. The third technique is a simplification of the CRI, while the first two involve other schemes to implement the GMP.

Closed systems often expose a number of interesting characteristics:

1. The membership functions involved can be efficiently stored in memory by means of a small number of parameters.
2. The membership function resulting from the inference process can be immediately linguistically interpreted.
3. The computational overhead for the CRI mentioned above can be avoided because the inference is essentially restricted to simple parameter manipulation.

The last two arguments make these techniques extremely suitable for inference applications with linguistic inputs and outputs.

4. HELLENDOORN APPROACH

4.1 GENERAL IDEA

In [6] Hellendoorn presents a technique for implementing the GMP, that can be applied when A, B, A' and B' in (SCHEME–2) are fuzzy sets with increasing membership functions. It is assumed that they all have the same shape. Furthermore the membership function for B' depends on the center and the length of its shell. These can be calculated on the basis of the shells of A, B and A'. In this inference process the CRI is not used.

Before we go into the details, we discuss an older proposal Hellendoorn advanced towards a system closed for inference. As opposed to the newer one, this older approach is based on the CRI. We will show that it is not suitable to model inference using IF–THEN rules.

4.2 CLASS OF MEMBERSHIP FUNCTIONS

The universe will be an interval of $\mathbb{R}$, i.e., $[a,b]$ for some $a \in \mathbb{R}, b \in \mathbb{R}, a \leq b$. For such an interval, we define a particular set of quadruples: $\mathbb{P}_{[a,b]} = \{(\alpha,\beta,\gamma,\delta) | (\alpha,\beta,\gamma,\delta) \in [a,b]^4$ and $\alpha \leq \beta \leq \gamma \leq \delta$ and ($\alpha < \beta$ or $\alpha = \beta = a$) and ($\gamma < \delta$ or $\gamma = \delta = b$)$\}$ Each quadruple consists of 4 parameters characterizing a membership function.

Definition 6 (ϕ–generator, ϕ–function, Γ–function) *Let ϕ_1 and ϕ_2 be continuous $[0,1] - [0,1]$ mappings, ϕ_1 increasing and ϕ_2 decreasing, with boundary conditions $\phi_1(0) = 0$, $\phi_1(1) = 1$, $\phi_2(0) = 1$ and $\phi_2(1) = 0$. Let $[a,b]$ be an interval of $\mathbb{R}$. For $(\alpha, \beta, \gamma, \delta)$ in $\mathbb{P}_{[a,b]}$, the $[a,b] - [0,1]$ mapping $\phi_{\phi_1,\phi_2}(.;\alpha,\beta,\gamma,\delta)$ is called a ϕ-function and is defined as (for x in $[a,b]$):*

$$\phi_{\phi_1,\phi_2}(x;\alpha,\beta,\gamma,\delta) = \begin{cases} 0, & \text{for } x \in [a,\alpha] \quad \text{if } \alpha < \beta \\ \phi_1(\frac{x-\alpha}{\beta-\alpha}), & \text{for } x \in [\alpha,\beta] \quad \text{if } \alpha < \beta \\ 1, & \text{for } x \in [\beta,\gamma] \\ \phi_2(\frac{x-\gamma}{\delta-\gamma}), & \text{for } x \in [\gamma,\delta] \quad \text{if } \gamma < \delta \\ 0, & \text{for } x \in [\delta,b] \quad \text{if } \gamma < \delta \end{cases}$$

ϕ_{ϕ_1,ϕ_2} is called a ϕ-generator. $\phi_{\phi_1,\phi_2}(.;\alpha,\beta,b,b)$ is an increasing function. It is called a Γ-function and denoted $\Gamma_{\phi_1}(.;\alpha,\beta)$.

4.3 A CRI BASED CLOSED SYSTEM

For a particular class of fuzzy relations R, the application of $cri^{\mathcal{T}_M}_R$ to a ϕ-function results in another ϕ-function.

Lemma 1 *[5] Let $(\alpha_1,\beta_1,\gamma_1,\delta_1) \in \mathbb{P}_{[a_1,b_1]}$, $(\alpha_2,\beta_2,\gamma_2,\delta_2) \in \mathbb{P}_{[a_2,b_2]}$ and D the $\mathbb{R}^2 - \mathbb{R}$ mapping defined by $(\forall(x,y) \in \mathbb{R}^2)(D(x,y) = y - x)$.*

If $\quad R = \phi_{\phi_1,\phi_2}(.;\alpha_2,\beta_2,\gamma_2,\delta_2) \circ D$
$\quad A' = \phi_{\phi_1,\phi_2}(.;\alpha_1,\beta_1,\gamma_1,\delta_1)$
$\quad B' = cri^{\mathcal{T}_M}_R(A')$

Then $B' = \phi_{\phi_1,\phi_2}(.;\alpha_1+\alpha_2,\beta_1+\beta_2,\gamma_1+\gamma_2,\delta_1+\delta_2)$

For the proof we refer to [5]. We remark that R is a fuzzy relation on $[a_1,b_1]\times[a_3,b_3]$, with $[a_1,b_1]$ being the universe of A' and $[a_3,b_3]$ such that $(\forall(x,y)\in\mathbb{R}^2)\ ((x,y)\in[a_1,b_1]\times[a_3,b_3] \Leftrightarrow y-x\in[a_2,b_2])$, i.e., $a_3=a_2+b_1$ and $b_3=a_1+b_2$. Hence while A' is a fuzzy set on $[a_1,b_1]$, B' will be a fuzzy set on $[a_3,b_3]$.

The following theorem is straightforward from this lemma:

Theorem 2 *Let* $(\alpha_2,\beta_2,\gamma_2,\delta_2)\in\mathbb{P}_{[a_2,b_2]}$,

$$R = \phi_{\phi_1,\phi_2}(.;\alpha_2,\beta_2,\gamma_2,\delta_2)\circ D$$

$$U = \{\phi_{\phi_1,\phi_2}(.;\alpha_1,\beta_1,\gamma_1,\delta_1)|(\alpha_1,\beta_1,\gamma_1,\delta_1)\in\mathbb{P}_{[a_1,b_1]}\}$$

$$V = \{\phi_{\phi_1,\phi_2}(.;\alpha_1+\alpha_2,\beta_1+\beta_2,\gamma_1+\gamma_2,\delta_1+\delta_2)|(\alpha_1,\beta_1,\gamma_1,\delta_1)\in\mathbb{P}_{[a_1,b_1]}\}$$

Then $(U,V,cri_R^{\mathcal{T}_M})$ *is a closed system.*

In other words applying the CRI, we get the following inference scheme:

$$\frac{\begin{array}{l}\text{x is } \phi_{\phi_1,\phi_2}(.;\alpha_1,\beta_1,\gamma_1,\delta_1)\\ \text{x and y have relation } \phi_{\phi_1,\phi_2}(.;\alpha_2,\beta_2,\gamma_2,\delta_2)\circ D\end{array}}{\text{y is } \phi_{\phi_1,\phi_2}(.;\alpha_1+\alpha_2,\beta_1+\beta_2,\gamma_1+\gamma_2,\delta_1+\delta_2)} \qquad \text{(SCHEME-3)}$$

This works well for relations that are fuzzy comparators like *"is taller than," "is approximately equal to,"*... The following theorem, however, indicates that there is no straightforward correspondence between the usual representation of an IF–THEN rule by means of an implicator $\mathcal{I}$, and the representation of a fuzzy relation by means of a ϕ-function as done above.

Theorem 3 *Let* $a \in]0,+\infty[$. *There is no quadruple* (e,f,g,h) *in* $\mathbb{R}^4$ *such that for all* x *and* y *in* $[a,4a]$ *:*

$$\mathcal{I}(\Gamma_{\phi_1}(x;2a,3a),\Gamma_{\phi_1}(y;2a,3a)) = \phi_{\phi_1,\phi_2}(y-x;e,f,g,h)$$

Proof Suppose that we could find such an e,f,g and h, then from $\phi_{\phi_1,\phi_2}(2a-3a;e,f,g,h)=\phi_{\phi_1,\phi_2}((2a+a)-(3a+a);e,f,g,h)$ we would derive $\mathcal{I}(\Gamma_{\phi_1}(3a;2a,3a),\ \Gamma_{\phi_1}(2a;2a,3a))=\mathcal{I}(\Gamma_{\phi_1}(3a+a;2a,3a),\ \Gamma_{\phi_1}(2a+a;2a,3a))$. Hence $\mathcal{I}(1,0)=\mathcal{I}(1,1)$, which conflicts with the definition of implicator. □

4.4 A SHELL–BASED APPROACH

In [6] Hellendoorn presents another implementation of the GMP which can be used if all the membership functions involved in (SCHEME–2) belong to a particular class of increasing functions. In this section we will only consider Γ–functions defined by means of one and the same $[0,1]-[0,1]$ mapping ϕ_1 (defined as above in Definition 6). We will therefore drop ϕ_1 in the notation and denote them as $\Gamma(.;\alpha,\beta)$ for $\alpha\in X$, $\beta\in X$, $\alpha<\beta$, X a closed interval of the reals.

Let X and Y be closed intervals of the reals. For a,b,e and f in X, $a<b$, $e<f$ and c,d in Y, $c<d$: $\Gamma(.;a,b)$ and $\Gamma(.;e,f)$ are fuzzy sets on X, while $\Gamma(.;c,d)$ is a fuzzy set on Y. We want to make the inference:

$$\frac{\begin{array}{l}\text{x is } \Gamma(.;e,f)\\ \text{IF x is } \Gamma(.;a,b) \text{ THEN y is } \Gamma(.;c,d)\end{array}}{\text{y is } B'} \qquad \text{(SCHEME-4a)}$$

Hellendoorn states that B' is a Γ–function characterized by two parameters g and h $(g<h)$, i.e., $B'=\Gamma(.;g,h)$. According to him, g and h can be computed from a,b,c,d,e, and f in the following manner:

1. Compute the length l of the shell of B' by means of the lengths of the shells of the other three membership functions. It is assumed that the length of $shell(\Gamma(.;g,h))$ should be proportional to the length of $shell(\Gamma(.;c,d))$ as the length of $shell(\Gamma(.;e,f))$ is to the length of $shell(\Gamma(.;a,b))$. Therefore

$$l = h-g = \frac{(d-c)(f-e)}{b-a}$$

2. Compute the center z of the shell of B' by means of the measure of difference w between $\Gamma(.;a,b)$ and $\Gamma(.;e,f)$, i.e.,:

$$w = \frac{(a+b)-(e+f)}{(b-a)+(f-e)}$$

Hellendoorn proposes to calculate the center z as follows:

$$z = \frac{1}{2}((min(Y)-\frac{l}{2}-c)w^2+(min(Y)-\frac{l}{2}-d)w+c+d)$$

Remark that $w=0$ iff the centers of the shells of $\Gamma(.;a,b)$ and $\Gamma(.;e,f)$ coincide. In this case $z=\frac{c+d}{2}$ which means that the centers of the shells of $\Gamma(.;c,d)$ and B' will coincide as well. If $w=1$ then $a=f$, hence $\Gamma(.;a,b)\subseteq\Gamma(.;e,f)$. In other words

$\Gamma(.;e,f)$ is entirely weaker than $\Gamma(.;a,b)$, which means there is not enough information to make a meaningful derivation. In this case $z = min(Y) - \frac{l}{2}$, hence $(\forall y \in Y)(B'(y) = 1)$ which is usually interpreted as $B' = unknown$.

According to this definition of z, z does not necessarily belong to Y. If we want to develop a closed system, we can fix this by chosing $g = max(min(Y), z - \frac{l}{2})$ and $h = min(max(Y), z + \frac{l}{2})$.

Theorem 4 *Let X and Y be closed intervals of $\mathbb{R}$ and*

$$U = \{\Gamma(.;p,q) | \Gamma(.;p,q) \text{ is a } \Gamma\text{-function on } X\}$$

$$V = \{\Gamma(.;r,s) | \Gamma(.;r,s) \text{ is a } \Gamma\text{-function on } Y\}$$

Let a and b in X, $a < b$ and c and d in Y, $c < d$. Let the $U-V$ mapping $m_{(\Gamma(.;a,b),\Gamma(.;c,d))}$ be defined by (for e and f in X, $e < f$):

$$m_{(\Gamma(.;a,b),\Gamma(.;c,d))}(\Gamma(.;e,f)) = \Gamma(.;g,h)$$

with
$$\begin{aligned} g &= max(min(Y), z - \tfrac{l}{2}) \\ h &= min(max(Y), z + \tfrac{l}{2}) \\ z &= \tfrac{1}{2}((min(Y) - \tfrac{l}{2} - c)w^2 + (min(Y) - \tfrac{l}{2} - d)w + c + d) \\ w &= ((a+b) - (e+f))/((b-a) + (f-e)) \\ l &= ((d-c)(f-e))/(b-a) \end{aligned}$$

Then $(U, V, m_{(\Gamma(.;a,b),\Gamma(.;c,d))})$ is a closed system.

Substituting the proper Γ–functions in (SCHEME–2) we can make the following derivation:

x is $\Gamma(.;e,f)$
IF x is $\Gamma(.;a,b)$ THEN y is $\Gamma(.;c,d)$ (SCHEME-4b)

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y is $\Gamma(.; max(min(Y), z - \frac{l}{2}), min(max(Y), z + \frac{l}{2}))$

Example 1 *The $[0,1]-[0,1]$–mapping ϵ is defined by $\epsilon(x) = x$, for all x in $[0,1]$. Let $\phi_1 = \epsilon$. Consider the rule*

"IF speed is high THEN braking distance will be long" *(RULE–1)*

where "speed" and "braking distance" are variables that take values in the respective universes $X = [0,150]$ (expressed in km/h) and $Y = [0,300]$ (expressed in meters). The linguistic concept "high speed" *can be modelled by the fuzzy set $A = \Gamma(.;30,100)$ on X, while* "long braking distance" *can be represented by the fuzzy set $B = \Gamma(.;100,200)$ on Y. If speed is* "more or less high," *which can be modelled by $A' = \Gamma(.;10,90)$, then using (RULE–1) and (SCHEME–4b) we derive that braking distance is $B' = \Gamma(.;64, \frac{352}{7} + 128)$. Figure 1.1 depicts the membership functions for A, A', B and B'.*

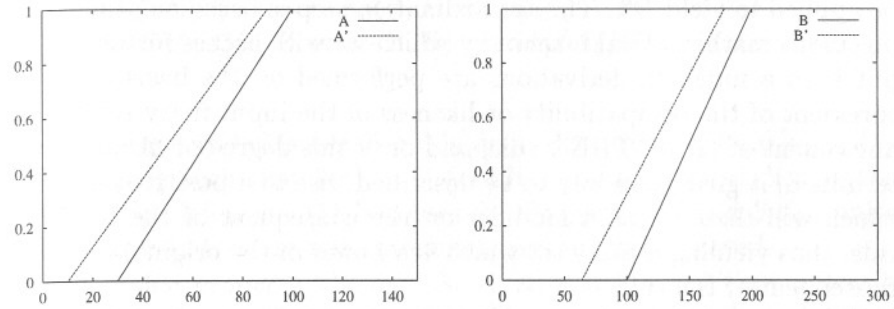

Figure 1.1 Hellendoorn approach a) Fuzzy sets on the universe of speeds b) Fuzzy sets on the universe of braking distances

5. NAFARIEH–KELLER APPROACH

5.1 GENERAL IDEA

The method of inference Nafarieh and Keller [7, 8] have proposed is specifically tailored to reasoning processes where linguistic hedges (i.e., adverbs such as *very, rather, quite, ...*) are modelled by real powers of a given base fuzzy set; we call them **powering hedges**. The general idea is to restate the GMP as (n represents a positive real number)

x is A^n
IF x is A THEN y is B (SCHEME–5)

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y is B^n

This corresponds to a hedge–preserving semantics of a fuzzy rule: when x is $m(A)$, then y should be $m(B)$ (where m represents a hedge). One should however be very cautious when adopting such an interpretation of a rule, as its use is not always justified by the context. Consider, for instance, the rule: "IF it is warm THEN it is pleasant to go out for a walk." Modelling this rule using the above strategy seems unrealistic, for it would lead us to believe that the most pleasant walks can be taken on the surface of the sun.

In [2] it is argued that a requirement like hedge-preservation underlies that the rule expresses more than a simple logical implication, for in classical logic it holds that a proposition inferred by modus ponens can never be more specific than what the consequent of the rule says.

The case where the input fuzzy set A' in (SCHEME–2), is not expressed as a real power of A, may be adequately handled by approximating A' by A^n for a certain n in order that the foregoing scheme can be applied to yield B^n. The approximation step is based on some rather ingenious mathematical machinery which we will discuss further on. To put it in a nutshell, derivations are performed on the basis of a measurement of the compatibility or likeness of the input fuzzy set and the antecedent of an IF–THEN rule, and once this degree of likeness is determined, it gives, in a way to be described, rise to a positive real power which will then act as a modifier on the consequent of the IF–THEN rule, thus yielding a fuzzy set which is a power of the original one in the consequent of the rule.

Nafarieh and Keller have applied this mechanism to control an automatic target recognizer. Their results are described in [8].

5.2 CLASS OF MEMBERSHIP FUNCTIONS

In order to simplify our discussion somewhat, we shall only consider intervals of $\mathbb{R}$ as candidate universes, i.e. $X = [u_1, u_2]$, $u_1 \in \mathbb{R}$, $u_2 \in \mathbb{R}$, $u_1 < u_2$. Within such a universe we will consider the class of convex fuzzy sets with a continuous membership function and an infinite support. We will denote this class of fuzzy sets on $[u_1, u_2]$ for which the Nafarieh–Keller approach is suitable, as $\mathcal{NK}([u_1, u_2])$. The convexity requirement seems quite natural for fuzzy sets representing linguistic concepts as *"old," "very hot," "about five and a half."* Demanding our fuzzy sets to have an infinite support actually rules out crisp numbers, which may seem a bit harsh at first, yet, as was stated above, simpler and more efficient techniques have been conceived in those circumstances where a crisp input is available.

5.3 MODE OF OPERATION

First we introduce a measure of compatibility between two fuzzy sets A and B belonging to the class $\mathcal{NK}([u_1, u_2])$.

Definition 7 (compatibility index) *The compatibility index* comp *is defined as the* $\mathcal{NK}([u_1, u_2]) - [0,1]$ *mapping defined by (for* A *and* B *in* $\mathcal{NK}([u_1, u_2])$*):*

$$\mathsf{comp}(A,B) = \frac{\int_{u_1}^{u_2}(A \cap B)(x)dx}{\int_{u_1}^{u_2}(A \cup B)(x)dx}$$

Proposition 1 *For* A, A_1 *and* B *in* $\mathcal{NK}([u_1, u_2])$*:*

1. $A \cap B = \emptyset \Rightarrow \mathsf{comp}(A,B) = 0$
2. $\mathsf{comp}(A,A) = 1$ *(reflexivity)*
3. $\mathsf{comp}(A,B) = \mathsf{comp}(B,A)$ *(symmetry)*
4. $A \subseteq A_1 \subseteq B \Rightarrow \mathsf{comp}(A,B) \leq \mathsf{comp}(A_1,B)$
5. $B \subseteq A_1 \subseteq A \Rightarrow \mathsf{comp}(A,B) \leq \mathsf{comp}(A_1,B)$

Proof.
The first property holds because $\emptyset$ is defined as the constant $[u_1, u_2] - [0,1]$ mapping such that $\emptyset(x) = 0$, for all x in $[u_1, u_2]$ and, hence, the numerator in definition (7) becomes 0. Properties 2 and 3 are quite obvious, so only the last two statements remain to be proved.

To show that property 4 holds, take A, A_1 and B in $\mathcal{NK}([u_1, u_2])$ such that $A \subseteq A_1 \subseteq B$. This means that $(\forall x \in [u_1, u_2])(A(u) \leq A_1(u) \leq B(u))$, and hence we have, due to the monotonicity of the *min* operator, for all $u \in [u_1, u_2]$ that $min(A(u), B(u)) \leq min(A_1(u), B(u))$. In terms of functions, this is equivalent to saying that $min \circ (A_1, B)$ dominates $min \circ (A, B)$ on $[u_1, u_2]$ and hence due to the monotonicity of integration we obtain $\int_{u_1}^{u_2}(A \cap B)(x)dx \leq \int_{u_1}^{u_2}(A_1 \cap B)(x)dx$. It is also easy to see that $\forall u \in [u_1, u_2], max(A(u), B(u)) = max(A_1(u), B(u)) = B(u)$, so finally we may conclude that $\mathsf{comp}(A,B) \leq \mathsf{comp}(A_1,B)$. Property 5 may be proved analogously. □

Definition 8 (reference fuzzy set) *Given a fuzzy set* $A \in \mathcal{NK}([u_1, u_2])$*, we introduce a reference fuzzy set*[2] *on the unit interval,* ref_A*. To define* ref_A *we first construct the linear mapping* h*:*

$$h: \begin{array}{ccc} [u_1, u_2] & \rightarrow & [0,1] \\ x & \mapsto & \frac{x-u_1}{u_2-u_1}, \quad \forall x \in [u_1, u_2] \end{array}$$

ref_A *is then defined as:*

$$\mathsf{ref}_A: \begin{array}{ccc} [0,1] & \rightarrow & [0,1] \\ x & \mapsto & A(h^{-1}(x)), \quad \forall x \in [0,1] \end{array}$$

Theorem 5 *Let the universe* $[u_1, u_2]$ *be an interval of* $\mathbb{R}$*,* $u_1 < u_2$*,* $A \in \mathcal{NK}([u_1, u_2])$ *and* ref_A *its reference set. Then we have the following equality for any* $n \in [0, +\infty[$*:*

$$\frac{\int_0^1 \mathsf{ref}_A^n(x)dx}{\int_0^1 \mathsf{ref}_A(x)dx} = \frac{\int_{u_1}^{u_2} A^n(x)dx}{\int_{u_1}^{u_2} A(x)dx} \qquad (1.1)$$

[2] Nafarieh and Keller call it the fuzzy truth value "true."

Proof.

The equality in (1.1) can be shown by applying a simple substitution to the integrands in the numerator and the denominator of the left hand side. Let $u = h^{-1}(x) = (u_2 - u_1)x + u_1$, so that $du = (u_2 - u_1)dx$. We easily get

$$\frac{\int_0^1 (A \circ h^{-1})^n(x)dx}{\int_0^1 (A \circ h^{-1})(x)dx} = \frac{\int_{u_1}^{u_2} \frac{A^n(u)}{u_2-u_1}du}{\int_{u_1}^{u_2} \frac{A(u)}{u_2-u_1}du}, \tag{1.2}$$

which is equivalent to the right hand side of (1.1), since $u_1 \neq u_2$. □

Due to this result, we are now in a position to establish a remarkable link between the previously defined compatibility index comp and the reference set ref_A. Indeed, since

$$\frac{\int_{u_1}^{u_2} A^n(x)dx}{\int_{u_1}^{u_2} A(x)dx} = \begin{cases} \frac{\int_{u_1}^{u_2}(A \cap A^n)(x)dx}{\int_{u_1}^{u_2}(A \cup A^n)(x)dx} = \mathsf{comp}(A, A^n) & \text{for } n \geq 1 \\ \frac{\int_{u_1}^{u_2}(A \cup A^n)(x)dx}{\int_{u_1}^{u_2}(A \cap A^n)(x)dx} = \frac{1}{\mathsf{comp}(A,A^n)} & \text{for } n < 1 \end{cases} \tag{1.3}$$

we have

$$\int_0^1 \mathsf{ref}_A^n(x)dx = \begin{cases} \left(\int_0^1 \mathsf{ref}_A(x)dx\right) \mathsf{comp}(A, A^n) & \text{for } n \geq 1 \\ \frac{\int_0^1 \mathsf{ref}_A(x)dx}{\mathsf{comp}(A,A^n)} & \text{for } n < 1 \end{cases} \tag{1.4}$$

In case the input A' in (SCHEME–2) is not expressed as a power of A, then (1.4) can still be used to approximate the degree that A' matches A. For instance, when $A' \subset A$, we could try to find a real value $n > 1$ such that $\mathsf{comp}(A', A)$ is approximately equal to $\mathsf{comp}(A^n, A)$. We do this by changing $\mathsf{comp}(A, A^n)$ in (1.4) by $\mathsf{comp}(A, A')$ and solving the resulting equation in the unknown n. A similar argument can be made when $A \subset A'$, to the effect that an $n < 1$ should be searched such that $\mathsf{comp}(A', A)$ is approximately equal to $\mathsf{comp}(A^n, A)$. In the remaining case (i.e., A' is incomparable to A w.r.t. $\subseteq$), A' can be considered less specific than A for the inference result, so we should still look for an $n < 1$ (we want the result of the inference step to be weaker than the consequent of the rule). From (1.4) one can easily deduce that the minimum value of $\mathsf{comp}(A, A^n)$ when $n < 1$ is arrived at when $n = 0$, namely

$$\mathsf{comp}^* = \mathsf{comp}(A, A^0) = \frac{\int_0^1 \mathsf{ref}_A(x)dx}{\int_0^1 \mathsf{ref}_A^0(x)dx} = \int_0^1 \mathsf{ref}_A(x)dx \tag{1.5}$$

When $\mathsf{comp}(A', A) < \mathsf{comp}^*$ (for example when A' and A are disjoint), there will be no solution to the equation (1.4). Nafarieh and Keller suggest to set $n = 0$ in that case. The derived fuzzy set B^n will correspond to the universe. This can be taken as a represention of total ignorance ($B^n = \mathit{unknown}$), and is in accordance with our intuition, because when the input and antecedent differ too much it should not be possible to infer anything with the GMP.

Theorem 6 *Let $[u_1, u_2]$ and $[v_1, v_2]$ be intervals of $\mathbb{R}$, $u_1 < u_2$, $v_1 < v_2$. Let $A \in \mathcal{NK}([u_1, u_2])$ and $B \in \mathcal{NK}([v_1, v_2])$. Let the $\mathcal{NK}([u_1, u_2]) - \mathcal{NK}([v_1, v_2])$ mapping $m_{(A,B)}$ be defined by (for $A' \in \mathcal{NK}([u_1, u_2])$):*

$$m_{(A,B)}(A') = B^n$$

with n defined as follows :

Let (a) : $\int_0^1 \mathsf{ref}_A^n(x)dx = \mathsf{comp}(A, A') \int_0^1 \mathsf{ref}_A(x)dx$

Let (b) : $\mathsf{comp}(A, A') \int_0^1 \mathsf{ref}_A^n(x)dx = \int_0^1 \mathsf{ref}_A(x)dx$

if $A' \subseteq A$ then
if (a) has a solution n_0 and $n_0 \geq 1$ then $n := n_0$
else $n := 0$
else if (b) has a solution n_0 and $n_0 < 1$ then $n := n_0$
else $n := 0$

Then $(\mathcal{NK}([u_1, u_2]), \{B^n | B \in \mathcal{NK}([v_1, v_2]), n \in [0, +\infty[\}, m_{(A,B)})$ is a closed system.

Example 2 *Let us consider (RULE–1) and fuzzy sets A and B from example 1 again. In this section we will also use the $[0,1]-[0,1]$ mapping θ defined by $\theta(x) = 1 - x$, for all x in $[0,1]$. When we receive an input fuzzy set A', we have to distinguish between two cases. If the input is a positive real power n of the antecedent, then there is no need to perform any calculations at all and we immediately acquire the result, B^n. This* *is especially so when the input exactly matches the antecedent, so at that stage we retrieve the (highly desirable!) coincidence with the modus ponens (SCHEME-1).*

Let us concentrate now on what happens when A' is not expressed as a power of A. Our first step will then be to construct the reference set of A. It can be verified that $\mathsf{ref_A} = \Gamma_\epsilon(.; \frac{1}{5}, \frac{2}{3})$. *Next, we have to calculate the integral*

$$\int_0^1 \mathsf{ref}_A^n(x)dx, n \in [0, +\infty[$$

It is worth noticing that these calculations are independent of A' and thus have to be performed just once. We find that

$$\int_0^1 \mathsf{ref}_A^n(x)dx = \int_{\frac{1}{5}}^{\frac{2}{3}} \left(\frac{15x-3}{7}\right)^n dx + \int_{\frac{2}{3}}^{1} dx = \frac{7}{15(n+1)} + \frac{1}{3} \qquad (1.6)$$

We now consider some specific inputs A' for the linguistic variable "speed." It's interesting to investigate what happens in case **shifting hedges**, *that modify linguistic terms by applying a shift rather than an involution to the associated membership function, are used. For instance, consider* $A' = \Gamma_\epsilon(.; 40, 110)$. *The first step in the inference process is to assess the measure of compatibility* $\mathsf{comp}(A, A')$ *between the antecedent and the input fuzzy set. Using Definition (7), we achieve*

$$\begin{aligned} \mathsf{comp}(A, A') &= \frac{\int_0^{150} min(A(x), A'(x))dx}{\int_0^{150} max(A(x), A'(x))dx} \\ &= \frac{\int_{40}^{110} \frac{x-40}{70}dx + \int_{110}^{150} dx}{\int_{30}^{100} \frac{x-30}{70}dx + \int_{100}^{150} dx} = \frac{15}{17} \end{aligned}$$

Since $A' \subset A$, we have to look for an $n \geq 1$ that satisfies equation (a) in theorem 6. From $\frac{7}{15(n+1)} + \frac{1}{3} = \frac{15}{17}\frac{17}{30}$ *we find* $n = \frac{9}{5}$. *The output fuzzy set* $B' = B^n$ *is then* $(\Gamma_\epsilon(.; 100, 200))^{\frac{9}{5}}$. *The fuzzy sets A, A', B and B' are depicted in Figure 1.2.*

We should remark that our result does not fully correspond to the hedge-preserving semantics we have outlined in the previous section, since this would mean that the fuzzy set B should be shifted in exactly the same way (generally undesirable when antecedent and consequent are defined on different universes). On the other hand, the input is somewhat stronger than the antecedent, and likewise the output is somewhat

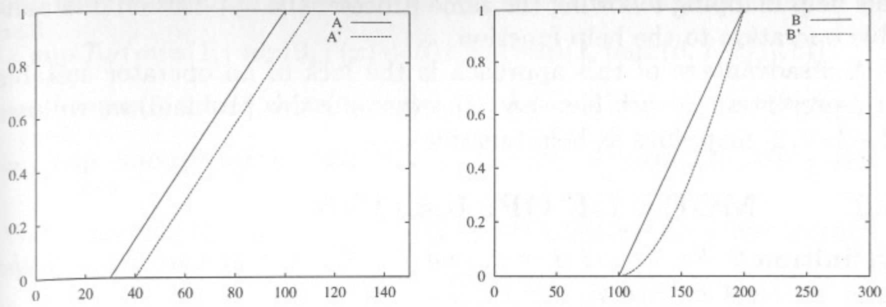

Figure 1.2 Nafarieh–Keller approach a) Fuzzy sets on the universe of speeds b) Fuzzy sets on the universe of braking distances

stronger than the consequent of the rule, which sounds intuitively correct. The hard part in using shifted hedges will be to devise suitable linguistic expressions for characterizing the output.

Finally let us consider the concept "around one hundred" *represented by* $A' = \phi_{\epsilon,\theta}(.; 90, 100, 100, 110)$. *Remark that $A' \subset A$. We first compute* $\mathsf{comp}(A, A')$:

$$\begin{aligned} \mathsf{comp}(A, A') &= \frac{\int_0^{150} min(A(x), A'(x))dx}{\int_0^{150} max(A(x), A'(x))dx} \\ &= \frac{\int_{90}^{100} \frac{x-90}{10}dx + \int_{100}^{110} \frac{110-x}{10}dx}{\int_{30}^{100} \frac{x-30}{70}dx + \int_{100}^{150} dx} = \frac{2}{17} \end{aligned}$$

From $\frac{7}{15(n+1)} + \frac{1}{3} = \frac{2}{17}\frac{17}{30}$ *we find* $n = \frac{-11}{4}$. *Hence there is no $n \geq 1$ which satisfies equation (a) in theorem 6 and we set $n = 0$, which means that the inferred result is* unknown. *The reason for this is that A and A' are not sufficiently similar.*

6. BOUCHON APPROACH

6.1 GENERAL IDEA

In [3] Bouchon presents a closed system in which the inference is based on the CRI. The key to her approach is the use of operators representing linguistic hedges (approximately, rather, about, almost). Unlike those of other authors, such an adverb representing operator does not act on a fuzzy set on $\mathbb{R}$ (i.e., a $\mathbb{R}-[0,1]$ mapping), but on a $\mathbb{R}-]-\infty, 1]$ mapping, which serves as a help mapping. The fuzzy sets representing a <term>, and <rather term>, <approximately term>,... are all constructed from this help mapping following the same process: the application of a suitable operation to the help function.

A disadvantage of this approach is the lack of an operator suitable to represent an adverb like very. To overcome this problem, we will use $\mathbb{R} - [-2,2]$ mappings as help functions.

6.2 MODE OF OPERATION

Definition 9 *For α and β in $\mathbb{R}$ and f a $\mathbb{R}-\mathbb{R}$ mapping, $f_{(\alpha,\beta)}$ is the fuzzy set on $\mathbb{R}$ defined by (for $x \in \mathbb{R}$):*

$$f_{(\alpha,\beta)}(x) = min(1, max(0, \alpha f(x) + \beta))$$

Proposition 2 *For $\beta \in [0,1]$, f a $\mathbb{R}-\mathbb{R}$ mapping:*

(1) $f_{(1,-\beta)} \subseteq f_{(1,0)}$
(2) $f_{(1,0)} \subseteq f_{(1-\beta,\beta)}$
(3) $f_{(1,0)} \subseteq f_{(1,\beta)}$

Proof Let $x \in \mathbb{R}$. The inequalities $f(x) - \beta \leq f(x)$ and $f(x) \leq f(x) + \beta$ are quite trivial. Adding the fact that min and max are increasing proves the first and the last inclusion. To prove the middle one, we remark that for $f(x) \leq 1$, $f(x) \leq (1-\beta)f(x) + \beta$. If $f(x) > 1$ then both $f_{(1,0)}(x)$ and $f_{(1-\beta,\beta)}(x)$ are 1. □

If the fuzzy set $f_{(1,0)}$ represents a <term>, then for a particular choice of $\beta \in [0,1]$, $f_{(1,-\beta)}$ can represent <very term>, while $f_{(1-\beta,\beta)}$ and $f_{(1,\beta)}$ can model <rather term> and <more or less term> respectively.

We first formulate a lemma that will help us to state under which conditions this generalized Bouchon approach for modelling hedges, gives rise to a closed system for inference.

Lemma 2 *For f a $\mathbb{R}-\mathbb{R}$ mapping, $wd(f) = [-2,2]$, $b \in [0,1]$, $\beta \in [0,1]$:*

(i) $\sup_{x \in \mathbb{R}} \mathcal{T}_W(f_{(1,\beta)}(x), \mathcal{I}_R(f_{(1,0)}(x), b)) = (1-\beta)b + \beta$
(ii) $\sup_{x \in \mathbb{R}} \mathcal{T}_W(f_{(1,-\beta)}(x), \mathcal{I}_R(f_{(1,0)}(x), b)) = b$
(iii) $\sup_{x \in \mathbb{R}} \mathcal{T}_W(f_{(1-\beta,\beta)}(x), \mathcal{I}_R(f_{(1,0)}(x), b)) = max(b, \beta)$

Proof As an example we prove *(i)*.

$$\begin{aligned}
&\sup_{x \in \mathbb{R}} \mathcal{T}_W(f_{(1,\beta)}(x), \mathcal{I}_R(f_{(1,0)}(x), b)) \\
&= \sup_{x \in \mathbb{R}} \mathcal{T}_W(min(1, max(0, f(x) + \beta)), \mathcal{I}_R(min(1, max(0, f(x))), b)) \\
&= \sup_{x \in \mathbb{R}} max(0, min(1, max(0, f(x) + \beta)) + (b-1)min(1, max(0, f(x)))) \\
&= \sup_{z \in wd(f)} max(0, min(1, max(0, z + \beta)) + (b-1)min(1, max(0, z)))
\end{aligned}$$

We divide $wd(f)$ into 5 intervals and calculate the supremum in each of them :

1. $X_1 = [-2, -\beta[$
   $$\begin{aligned}&\sup_{z \in X_1} max(0, min(1, max(0, z + \beta)) + (b-1)min(1, max(0, z))) \\ &= \sup_{z \in X_1} max(0, min(1, 0) + (b-1)min(1, 0))) = 0\end{aligned}$$

2. $X_2 = [-\beta, 0]$
   $$\begin{aligned}&\sup_{z \in X_2} max(0, min(1, max(0, z + \beta)) + (b-1)min(1, max(0, z))) \\ &= \sup_{z \in X_2} max(0, min(1, z + \beta) + (b-1)min(1, 0))) \\ &= \sup_{z \in X_2} max(0, min(1, z + \beta)) \\ &= \sup_{z \in X_2} min(1, z + \beta) = min(1, \beta) = \beta\end{aligned}$$

3. $X_3 =]0, 1-\beta]$
   $$\begin{aligned}&\sup_{z \in X_3} max(0, min(1, max(0, z + \beta)) + (b-1)min(1, max(0, z))) \\ &= \sup_{z \in X_3} max(0, z + \beta + (b-1)z) \\ &= max(0, (1-\beta)b + \beta) = (1-\beta)b + \beta\end{aligned}$$

4. $X_4 =]1-\beta, 1]$
   $$\begin{aligned}&\sup_{z \in X_4} max(0, min(1, max(0, z + \beta)) + (b-1)min(1, max(0, z))) \\ &= \sup_{z \in X_4} max(0, 1 + (b-1)z) \\ &= max(0, 1 + (b-1)(1-\beta)) \\ &= max(0, (1-\beta)b + \beta) = (1-\beta)b + \beta\end{aligned}$$

5. $X_5 =]1, 2]$
   $$\begin{aligned}&\sup_{z \in X_5} max(0, min(1, max(0, z + \beta)) + (b-1)min(1, max(0, z))) \\ &= max(0, 1 + (b-1)) = b\end{aligned}$$

Since $b \leq (1-\beta)b + \beta$ and $\beta \leq (1-\beta)b + \beta$:

$$max(0, \beta, (1-\beta)b + \beta, b) = (1-\beta)b + \beta$$

□

Theorem 7 *Let $\tilde{A}$ and $\tilde{B}$ be $\mathbb{R}-\mathbb{R}$ mappings, $wd(\tilde{A}) = wd(\tilde{B}) = [-2, 2]$, $\beta \in [0,1]$, k_β the constant $\mathbb{R} - \{k\}-$mapping,*

$$U = \{\tilde{A}_{(1,0)}, \tilde{A}_{(1,\beta)}, \tilde{A}_{(1,-\beta)}, \tilde{A}_{(1-\beta,\beta)}\}$$

$$V = \{\tilde{B}_{(1,0)}, \tilde{B}_{(1-\beta,\beta)} \cup k_\beta, \tilde{B}_{(1,0)} \cup k_\beta\}$$

$$m = cri^{\mathcal{T}_W}_{\tilde{A}_{(1,0)} \Rightarrow_{\mathcal{I}_R} \tilde{B}_{(1,0)}}$$

Then (U, V, m) is a closed system.

Proof Using (i) from Lemma 2, for all $y \in \mathbb{R}$:

$$\begin{aligned}
m(\tilde{A}_{(1,\beta)})(y) &= cri^{\mathcal{T}_W}_{\tilde{A}_{(1,0)} \Rightarrow_{\mathcal{I}_R} \tilde{B}_{(1,0)}}(\tilde{A}_{(1,\beta)})(y) \\
&= \sup_{x \in \mathbb{R}} \mathcal{T}_W(\tilde{A}_{(1,\beta)}(x), \mathcal{I}_R(\tilde{A}_{(1,0)}(x), \tilde{B}_{(1,0)}(y))) \\
&= (1-\beta)\tilde{B}_{(1,0)}(y) + \beta \\
&= (1-\beta)min(1, max(0, B(y))) + \beta \\
&= min(1, max(\beta, (1-\beta)B(y) + \beta)) \\
&= max(min(1, \beta), min(1, (1-\beta)B(y) + \beta)) \\
&= max(\beta, min(1, max(0, (1-\beta)B(y) + \beta))) \\
&= max(\beta, \tilde{B}_{(1-\beta,\beta)}(y))
\end{aligned}$$

Likewise we can deduct:

using (i): $m(\tilde{A}_{(1,\beta)}) = \tilde{B}_{(1,0)}$
using (ii): $m(\tilde{A}_{(1,-\beta)}) = \tilde{B}_{(1,0)}$
using (iii): $m(\tilde{A}_{(1-\beta,\beta)}) = \tilde{B}_{(1,0)} \cup k_\beta$

□

If in (SCHEME-2) $A = \tilde{A}_{(1,0)}$ and $B = \tilde{B}_{(1,0)}$, and the GMP is implemented by means of the CRI (using $\mathcal{T}_W, \mathcal{I}_R$) the following derivations can be made:

x is A	x is more or less A
IF x is A THEN y is B	IF x is A THEN y is B
y is B	y is rather B with uncertainty β

x is very A	x is rather A
IF x is A THEN y is B	IF x is A THEN y is B
y is B	y is B with uncertainty β

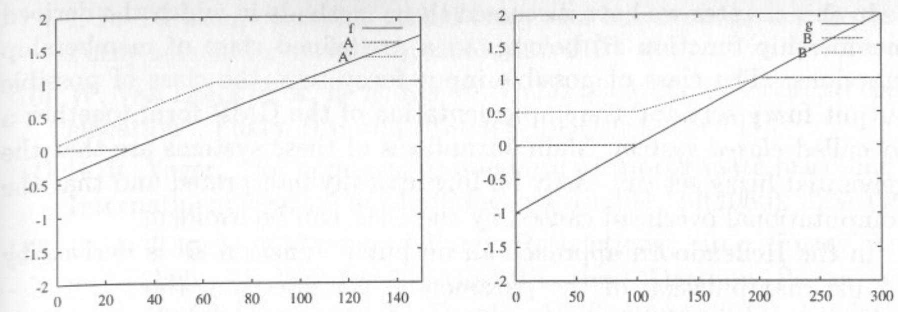

Figure 1.3 Generalized Bouchon approach a) Fuzzy sets on the universe of speeds b) Fuzzy sets on the universe of braking distances

Example 3 *We return to our car control example. Let f and g be defined by:*

$$f: \quad [0,150] \rightarrow [0,1], \qquad x \mapsto \frac{x-30}{70}, \quad \forall x \in [0,150]$$

$$g: \quad [0,300] \rightarrow [0,1], \qquad x \mapsto \frac{x-100}{100} \quad \forall x \in [0,300]$$

then $A = f_{(1,0)}$ represents "high speed" *while $B = g_{(1,0)}$ models* "long braking distance." *For $\beta = \frac{1}{2}$, $A' = f_{(1,\frac{1}{2})}$ represents* "more or less high speed." *We can make the derivation:*

speed is more or less high (A')
IF speed is high (A) THEN braking distance will be long (B)

---

braking distance is rather long with uncertainty $\frac{1}{2}$ $(B' = g_{(\frac{1}{2},\frac{1}{2})} \cup k_{\frac{1}{2}})$

The mappings f and g and the fuzzy sets A, A', B and B' are depicted in Figure 1.3.

7. CONCLUSION

The generalized modus ponens (GMP) is a useful deduction scheme in case the input "x is A'" does not exactly match the premise of the IF–THEN rule "IF x is A THEN y is B". In a fuzzy set theoretical framework, the GMP is usually implemented by means of the compositional rule of inference (CRI). However, if the input set A' is a fuzzy set (instead of a crisp singleton), the derivation of an output set B' involves large computational efforts. Furthermore the obtained membership function B' often still has to be interpreted by some process of linguistic approximation.

In this chapter we have discussed three methods in which the derived membership function B' belongs to a predefined class of membership functions. The class of possible input fuzzy sets, the class of possible output fuzzy sets and the implementation of the GMP form together a so–called *closed system*. Main advantages of these systems are that the generated fuzzy set can easily be linguistically interpreted and that the computational overhead caused by the CRI, can be avoided.

In the Hellendoorn approach an output Γ–function B' is derived by simple manipulation of the parameters characterizing the input Γ – function A' and the Γ–functions A and B involved in the IF–THEN rule. In the Nafarieh–Keller approach the output fuzzy set B' is a real power B^n of the fuzzy set B in the consequent of the IF–THEN rule. The power n is determined by the degree of compatibility between the input fuzzy set A' and the fuzzy set A in the premise of the rule. Finally the generalized Bouchon approach we have discussed, presents a shortcut to obtain the same results the CRI would give, by means of manipulation of parameters that correspond to linguistic hedges.

References

[1] J. F. Baldwin, "A new approach to approximate reasoning using a fuzzy logic", Fuzzy sets and systems, 2(1979), 309–325

[2] B. Bouchon-Meunier, D. Dubois, L. Godo, H. Prade, "Fuzzy sets and possibility theory in approximate and plausible reasoning", Fuzzy sets in approximate reasoning and information systems, Kluwer Academic Publishers, (1999), 15–190

[3] B. Bouchon–Meunier, "Stability of Linguistic Modifiers Compatible with a Fuzzy Logic", Lecture Notes in Computer Science 313, Proceedings IPMU '88, Springer–Verlag, (1988), 63–70

[4] M. R. Emami, I. B. Türksen, A. A. Goldenberg, "A unified parameterized formulation of reasoning in fuzzy modeling and control", Fuzzy sets and systems, 108(1999), 59–81

[5] H. Hellendoorn, "Closure Properties of the Compositional Rule of Inference", Fuzzy Sets and Systems, 35(1990), 163–183

[6] H. Hellendoorn, "The generalized modus ponens considered as a fuzzy relation", Fuzzy sets and systems, 46(1992), 29–48

[7] A. Nafarieh, J. M. Keller, "A New Approach to Inference in Approximate Reasoning", Fuzzy Sets and Systems, 41(1991), 17–37

[8] A. Nafarieh, J. M. Keller, "A Fuzzy Logic Rule–Based Automatic Target Recognizer", International Journal of Intelligent Systems, 6(1991), 295–312

[9] I. B. Türksen, "Interval valued fuzzy sets based on normal forms", Fuzzy sets and systems, 20(1986), 191–210

[10] W. Van Leeckwijck, E. E. Kerre, "Defuzzification: criteria and classification", Fuzzy Sets and Systems, 108(1999), 159–178

[11] R. R. Yager, "An approach to inference in approximate reasoning", International Journal of Man-Machine Studies, 13(1980), 323–338

[12] L. A. Zadeh, "Calculus of Fuzzy Restrictions", in : "Fuzzy Sets and Their Applications to Cognitive and Decision Processes", L. A. Zadeh, K.–S. Fu, K. Tanaka, M. Shimura (Ed.), Academic Press Inc. New York, (1975), 1–40