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Decision reducts and bireducts in a covering approximation space and their relationship to set definability

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ABSTRACT

In this paper, we discuss the relationship between different types of reduction and set definability. We recall the definition of a decision reduct, a γ-decision reduct, a decision bireduct and a γ-decision bireduct in a Pawlak approximation space and the notion of set definability both in a Pawlak and a covering approximation space. We extend the notion of discernibility between objects in a Pawlak approximation space to a covering approximation space. Moreover, we introduce the definition of a decision reduct, a γ-decision reduct, a decision bireduct and a γ-decision bireduct in a covering approximation space. In addition, we study interrelationships between the four types of reduction, the correspondence with positive regions and the relationship to set definability in Pawlak and covering approximation spaces.

1. Introduction

Rough set theory was developed by Pawlak [14] as a tool to analyze and reason about data. An important application of rough set theory in knowledge discovery is attribute subset selection [12]. There are different algorithms based on rough set theory which search for so-called decision reducts: irreducible sets of conditional attributes that satisfy predefined criteria for keeping enough information about decisions [18]. As the original definition of a reduct is quite restrictive, approaches concerning approximate reducts have been introduced (see e.g. [17]). Unfortunately, the current approximate reduct criteria are not optimal for the building of classifier ensembles and therefore, Ślęzak and Janusz [18] introduced the notion of decision bireducts inspired by the methodology of biclustering [13]. In this approach, both a subset of conditional attributes which describes the decision classes and a subset of objects of the universe for which such a description is valid are selected. In [19–21], Stawicki et al. discussed the differences and relationships between approximate reducts and decision bireducts. Moreover, they introduced the notions of a γ-decision reducts and bireducts, which correspond to the well-known notion of positive region in rough set decision systems.

The above notions of (γ-)decision reducts and bireducts are defined for Pawlak's rough set model. For each set of attributes, a corresponding equivalence relation can be constructed on the universe of discourse and thus, the universe can be partitioned. Such an equivalence relation describes whether two objects can be discerned given the information provided by the considered attributes. However, defining a partition is not always possible (see e.g. [7,8,10,11]). Instead, a covering of the universe related with a set of conditional attributes is constructed. Before extending the three types of reduction to covering approximation spaces, we first extend the notion of discernibility to covering approximation spaces.

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The indiscernibility class of an object will no longer be given by its equivalence class, but by its neighborhood [3]. Note that the neighborhood of an object is no longer necessarily symmetric, i.e., it is possible that an object x of the universe is discernible from an object y of the universe, but that y is not discernible from x.

The extension to covering approximation spaces is done in a semantically sound way. In [23], Yao argued that there are two sides of rough set theory: a computational and a conceptual or semantical one. The former focuses on how to compute concepts and on the construction of algorithms, while the latter studies how to define such concepts and provides insight in the concepts. In a semantical rough set approach, elementary and definable sets are defined. An elementary set is a set of objects which is directly related with the data and is constructed in a meaningful way. A definable set can be described by a union of elementary sets and can therefore be interpreted from the available knowledge given in the data. If a subset of objects is not definable, it can be approximated by definable sets [6].

In this article, we first investigate the connection between (γ-)decision reducts and bireducts, and set definability in Pawlak approximation spaces. In addition, we extend the notion of discernibility to covering approximation spaces. From this, we define decision reducts, γ-decision reducts, bireducts and γ-bireducts in a covering approximation space. Furthermore, we study which properties are maintained in the more general framework.

The remainder of this article is structured as follows. In Section 2, we discuss set definability in Pawlak and covering approximation spaces and the different types of reduction for Pawlak's rough set model. We recall the connection between a γ-decision bireduct and the positive region in a Pawlak approximation space. In Section 3, we study decision reducts, γ-decision reducts, bireducts and γ-decision bireducts and their connection with set definability in a Pawlak approximation space. The notion of discernibility and the four types of reduction are extended to covering approximation spaces in Section 4. Moreover, we study which properties of Section 2 and Section 3 are maintained. Section 5 presents a worked example to illustrate the practical meaning of the introduced concepts. To end, conclusions and future work are stated in Section 6.

2. Preliminaries

In this section, we discuss definability in Pawlak and covering approximation spaces. In addition, we recall the definitions of a decision reduct, a γ-decision reduct, a decision bireduct and a γ-decision bireduct for Pawlak's rough set model. All these notions rely on the concept of a decision system.

Definition 1. [15] A decision system $\mathbb{A}$ is a tuple $\mathbb{A} = (U, C \cup \{d\})$, where U is a non-empty set of objects called the universe of discourse, C is a non-empty set of conditional attributes and $d \notin C$ is the decision attribute. For every attribute $a \in C \cup \{d\}$ there is a non-empty set of attribute values V_a and a complete information function $I_a : U \to V_a$.

2.1. Definable sets in Pawlak and covering approximation spaces

An important topic in rough set theory is the granulation of the universe of discourse given the information provided in the data. Each granule is a subset of the universe U and represents a basic piece of knowledge [15,22]. In Pawlak's original rough set model, the family of granules is represented by a partition of the universe [14]: given a decision system $\mathbb{A}$ and a set of conditional attributes $B \subseteq C$, the equivalence relation E_B, also called the B-indiscernibility relation, is constructed as follows:

$$\forall x, y \in U : (x, y) \in E_B \Leftrightarrow \forall a \in B : I_a(x) = I_a(y).$$

The tuple (U, E_B) is called a Pawlak approximation space. The equivalence class of $x \in U$, denoted by $[x]_{E_B}$, consists of the objects which have the same attribute values for all the attributes in B. Hence, if $y \in [x]_{E_B}$, x and y are not discernible by the attributes of B. Therefore, the partition $U/E_B = \{[x]_{E_B} \mid x \in U\}$ can be seen as the family of basic granules given the set B, and each equivalence class $[x]_{E_B}$ is called an *elementary set* in (U, E_B) [5]. Moreover, the union of equivalence classes is called a *definable set* in (U, E_B), since such a set can be constructed and interpreted from the available knowledge given the set of attributes B [5]. By closing the partition U/E_B under set union, the family of definable sets is obtained:

$$\mathcal{B}(U/E_B) = \left\{ \bigcup F \mid F \subseteq U/E_B \right\}.$$

As U/E_B is a partition, $\mathcal{B}(U/E_B)$ is closed under set union, set intersection and set complement. Therefore, the family of definable sets in the Pawlak approximation space (U, E_B) is represented by the Boolean algebra $\mathcal{B}(U/E_B)$.

The definable sets can be used to approximate any $X \subseteq U$ through the associated lower and upper approximation operators:

$$\underline{\mathrm{apr}}_{E_B}(X) = \{x \in U \mid [x]_{E_B} \subseteq X\}, \overline{\mathrm{apr}}_{E_B}(X) = \{x \in U \mid [x]_{E_B} \cap X \neq \emptyset\}$$

It holds that $\underline{\mathrm{apr}}_{E_B}(X)$ is the maximal definable set contained by X, and $\overline{\mathrm{apr}}_{E_B}(X)$ is the minimal definable set containing X. [24]

Unfortunately, constructing an equivalence relation given a set of conditional attributes is not always possible or useful. For example, when there are missing values in the decision system [9–11], or when many equivalence classes only consist of one object [7,8]. When the discernibility relation is not represented by an equivalence relation, but e.g., by a tolerance or dominance relation, the family of basic granules is given by a covering of U instead of a partition.

Definition 2. [25] Given a universe U and an index set I, the collection $\mathbb{C} = \{K_i \subseteq U \mid K_i \neq \emptyset, i \in I\}$ is called a *covering* of U if $\bigcup_{i\in I} K_i = U$.

The construction of a covering related to one conditional attribute $a \in C$ should be done from a semantical point of view [5]. For example, the foresets of a dominance relation [7,8] or a tolerance relation [10,11] can be used to construct the covering $\mathbb{C}_a$: $\mathbb{C}_a = \{Rx \mid x \in U\}$ with R a meaningful binary relation related to the attribute values of a and Rx defined as follows: for $y \in U$, $y \in Rx$ if and only if $(y, x) \in R$. For a set of conditional attributes $B \subseteq C$, the covering $\mathbb{C}_B$ is defined as follows:

$$\mathbb{C}_B = \left\{ \bigcap_{a\in B} K_a \mid \bigcap_{a\in B} K_a \neq \emptyset, K_a \in \mathbb{C}_a, \forall a \in B \right\},$$

i.e., $\mathbb{C}_B$ is the set of non-empty intersections of elements from the coverings $\mathbb{C}_a$, $a \in B$ [5]. By convention, we define $\mathbb{C}_\emptyset = \{U\}$: when no attributes are considered, all objects are related to each other.

For $B \subseteq C$, the tuple $(U, \mathbb{C}_B)$ is called a covering approximation space. The basic granules are now given by the sets in the covering $\mathbb{C}_B$, called *patches*. Every patch $K \in \mathbb{C}_B$ is therefore called an elementary set in $(U, \mathbb{C}_B)$. Similarly as in Pawlak's rough set model, the family of definable sets in $(U, \mathbb{C}_B)$ is constructed by closing the covering $\mathbb{C}_B$ under set union [5]:

$$\cup^*(\mathbb{C}_B) = \left\{ \bigcup F \mid F \subseteq \mathbb{C}_B \right\}.$$

The set $\cup^*(\mathbb{C}_B)$ is called the *union-closure* of $\mathbb{C}_B$ and is closed under set union. However, as $\mathbb{C}_B$ is no longer a partition, $\cup^*(\mathbb{C}_B)$ is not closed under set intersection and set complement. We conclude that the family of definable sets in the covering approximation space $(U, \mathbb{C}_B)$ is given by the union-closure $\cup^*(\mathbb{C}_B)$.

Just as in Pawlak's framework, approximation operators can be used to approximate any set $X \in U$ by means of elements of $\cup^*(\mathbb{C}_B)$. Computational approaches to covering-based rough sets take various strategies to define suitable approximation operators starting from the covering $\mathbb{C}_B$. In this paper, we focus on the tight lower approximation [25,26] and its dual upper approximation:

$$\underline{\text{apr}}'_{\mathbb{C}_B}(X) = \bigcup \{K \in \mathbb{C}_B \mid K \subseteq X\}, \overline{\text{apr}}'_{\mathbb{C}_B}(X) = \text{co}(\underline{\text{apr}}'_{\mathbb{C}_B}(\text{co}X))$$

where co represents set-theoretic complement. The tight lower approximation operator appears to be the most interesting from conceptual point of view, as it is the unique maximal definable set contained by X, just like in Pawlak's rough set model. On the other hand, there is no unique minimal definable set containing X as opposed to Pawlak's model [6,24].

2.2. Decision reducts, γ-decision reducts, bireducts and γ-decision bireducts in a Pawlak approximation space

The concept of a decision reduct is a very important contribution of rough set theory. A key notion in determining a decision reduct is the notion of discernibility. In a Pawlak approximation space (U, E_B) with $B \subseteq C$, it is said that the objects $x, y \in U$ are discernible if $[x]_{E_B} \neq [y]_{E_B}$. This means that there is at least one attribute a in B such that $I_a(x) \neq I_a(y)$.

Decision reducts are normally considered only for consistent decision systems. A decision system $\mathbb{A}$ is *inconsistent* if there exist objects $x, y \in U$ with different decision values, i.e., $I_d(x) \neq I_d(y)$, which are not discernible in the Pawlak approximation space (U, E_C), i.e., $I_a(x) = I_a(y)$ for each $a \in C$. If this is not the case, then $\mathbb{A}$ is called *consistent*.

Definition 3. [15,20] Given the consistent decision system $\mathbb{A} = (U, C \cup \{d\})$, the set of conditional attributes $B \subseteq C$ is called a *decision reduct* for $\mathbb{A}$ if and only if each pair $x, y \in U$ satisfying the inequality $I_d(x) \neq I_d(y)$ is discerned in (U, E_B), and if B is irreducible, i.e., there is no smaller subset $B' \subsetneq B$ which satisfies this property. We denote the set of decision reducts by Red.

Note that $B = \emptyset$ is not excluded in the definition of a decision reduct. However, $B = \emptyset$ is only a decision reduct if all objects have the same decision, since $U/E_B = \{U\}$ when $B = \emptyset$.

In literature, more general definitions of decision reducts that apply also to inconsistent decision systems are sometimes used, based on the notion of the positive region. Let $\mathbb{A}$ be a decision system and $B \subseteq C$, the B-positive region $\text{Pos}(B)$ in Pawlak's rough set model is defined as the set of objects for which the values of B predict the decision class unequivocally:

$$\mathrm{Pos}(B) = \bigcup_{[x]_{E_d} \in U/E_d} \underline{\mathrm{apr}}_{E_B}([x]_{E_d}) = \bigcup_{[x]_{E_d} \in U/E_d} \{y \in U \mid [y]_{E_B} \subseteq [x]_{E_d}\} = \{x \in U \mid [x]_{E_B} \subseteq [x]_{E_d}\},$$

where E_d is the equivalence relation associated with the decision attribute d:

$$\forall x, y \in U : (x, y) \in E_d \Leftrightarrow I_d(x) = I_d(y).$$

Definition 4. [20] Given the decision system $\mathbb{A} = (U, C \cup \{d\})$, the set of conditional attributes $B \subseteq C$ is called a *γ-decision reduct* for $\mathbb{A}$ if and only if $\mathrm{Pos}(B) = \mathrm{Pos}(C)$, and if it is irreducible, i.e., there is no smaller subset $B' \subsetneq B$ which satisfies this property. We denote the set of γ-decision reducts by Red_γ.

For consistent decision systems, the notions of decision reduct and γ-decision reduct coincide, while for an inconsistent decision system $\mathbb{A}$, Stawicki et al. [20] discussed a procedure to transform $\mathbb{A}$ into a consistent decision system, such that its decision reducts are exactly the γ-decision reducts of the original decision system.

From the definition of a (γ-)decision reduct it is clear that it is obtained by selecting conditional attributes given predefined criteria, in order to keep enough information to make decisions [18]. Throughout this process, the set of objects U remains fixed. On the other hand, a decision bireduct also takes object selection into consideration, also commonly referred to as instance selection. Hence, both a set of attributes and a set of objects is selected. The former describes the decision classes and the latter contains the objects for which such a description is valid. Ślęzak and Janusz introduced the concept of decision bireducts for Pawlak's rough set model inspired by the methodology of biclustering [13].

Definition 5. [18] Let $\mathbb{A}$ be a decision system. A pair (B, X) with $B \subseteq C$ and $X \subseteq U$ is called a *decision bireduct* if and only if all pairs $x, y \in X$ with $I_d(x) \neq I_d(y)$ are discerned in the Pawlak approximation space (U, E_B) and the following properties hold:

1. There is no $B' \subsetneq B$ such that all pairs $x, y \in X$ with $I_d(x) \neq I_d(y)$ are discerned in $(U, E_{B'})$,
2. There is no $X' \supsetneq X$ such that all pairs $x, y \in X'$ with $I_d(x) \neq I_d(y)$ are discerned in (U, E_B).

For $B \subseteq C$, we denote $\mathbb{X}_B = \{X \subseteq U \mid (B, X) \text{ decision bireduct}\}$.

Hence, to obtain a decision bireduct (B, X) we select both conditional attributes and objects of the universe such that B is minimal for X and X is maximal for B. Note that the same set $B \subseteq C$ can occur as a component of many decision bireducts with different sets of objects and similarly, the set $X \subseteq U$ can occur as a component of many decision bireducts with different sets of conditional attributes. Note that for $B \subseteq C$ and $X = \emptyset$, (B, X) can never be a decision bireduct, as we can always extend X with any object $x \in U$.

An interesting question arises whether there is an optimal bireduct with the component B. An implicit assumption is that a decision bireduct (B, X) minimizes the set of conditional attributes as well as the set of outliers $U \setminus X$ [21]. The smaller the size of the set of outliers, the more general the description of the decision system. However, in case of imbalanced data sets, i.e., when there is a large disproportion between the different decision classes, the simplest form of measuring the size based on the cardinality of the object subsets may be insufficient. For example, assume there is one very large decision class $[x]_{E_d}$ and a couple of small decision classes $\{[y_1]_{E_d}, [y_2]_{E_d}, \ldots, [y_n]_{E_d}\}$. For $B \subseteq C$ and $X \in \mathbb{X}_B$, it is preferable that X does not only contain elements from the large decision class $[x]_{E_d}$, but also contains elements from the minority decision classes $[y_i]_{E_d}$. In such cases one should pay more attention to a specified subset of objects, for instance, objects belonging to the minority classes. With this in mind, Stawicki and Widz introduced the concept of a γ-decision bireduct for Pawlak's rough set model, where an object belongs to the component X if it can be discerned in the context of the whole universe U.

Definition 6. [21] Let $\mathbb{A}$ be a decision system. A pair (B, X) with $B \subseteq C$ and $X \subseteq U$ is called a *γ-decision bireduct* if and only if all pairs $x \in X, y \in U$ with $I_d(x) \neq I_d(y)$ are discerned in the Pawlak approximation space (U, E_B) and the following properties hold:

1. There is no $B' \subsetneq B$ such that all pairs $x \in X, y \in U$ with $I_d(x) \neq I_d(y)$ are discerned in $(U, E_{B'})$,
2. There is no $X' \supsetneq X$ such that all pairs $x \in X', y \in U$ with $I_d(x) \neq I_d(y)$ are discerned in (U, E_B).

For $B \subseteq C$, we denote $\mathbb{X}_B^\gamma = \{X \subseteq U \mid (B, X)\ \gamma\text{-decision bireduct}\}$.

Note that for $B \subseteq C$ it is possible that $\mathbb{X}_B^\gamma = \{\emptyset\}$, i.e., the empty set is the only subset of U which forms a γ-decision bireduct with the given set of attributes B. In comparison, $\mathbb{X}_B^\gamma = \emptyset$ means that B cannot operate as a component of a γ-decision bireduct, as it is not minimal.

Table 1
Decision system $\mathbb{A}$ from Example 1.

	a_1	a_2	d
x	1	1	0
y	1	1	1
z	0	1	1

Table 2
Decision system $\mathbb{A}$ from Example 2.

	a_1	a_2	d
x	1	0	0
y	1	1	1
z	0	1	1

In [21] it is stated that a set $X \in \mathbb{X}_B^\gamma$ is closely related with the positive region of B: an object can be added to X if it belongs to the positive region defined by B [21]. More formally, Stawicki et al. proved that $\text{Pos}(B)$ is the only possible set in $\mathbb{X}_B^\gamma$:

Theorem 1. *[20] Let $\mathbb{A}$ be a decision system and $B \subseteq C$. $\mathbb{X}_B^\gamma \neq \emptyset$ if and only if $\mathbb{X}_B^\gamma = \{\text{Pos}(B)\}$ and $\{\text{Pos}(B')\} \neq \{\text{Pos}(B)\}$ for all $B' \subset B$.*

Hence, given $B \subseteq C$, there is at most one γ-decision bireduct associated with B, namely $(B, \text{Pos}(B))$.

To end this section, we point out the following remark. If the decision system $\mathbb{A}$ is inconsistent, then there is no decision reduct according to Definition 4. However, it is possible to determine subsets $B, B' \subseteq C$ and subsets $X \subseteq U$ such that (B, X) is a decision bireduct according to Definition 5 and $(B', \text{Pos}(B'))$ is a γ-decision bireduct according to Definition 6. We illustrate this in the following example:

Example 1. Let $\mathbb{A}$ be the decision system with $U = \{x, y, z\}$ and $C = \{a_1, a_2\}$ presented in Table 1. The system $\mathbb{A}$ is inconsistent due to the objects x and y. Therefore, $\text{Red} = \emptyset$. However, we have the following decision bireducts and γ-decision bireducts:

$$\begin{array}{ll} \mathbb{X}_\emptyset = \emptyset & \mathbb{X}_\emptyset^\gamma = \emptyset \\ \mathbb{X}_{\{a_1\}} = \{\{x,z\},\{y,z\}\} & \mathbb{X}_{\{a_1\}}^\gamma = \{\{z\}\} \\ \mathbb{X}_{\{a_2\}} = \{\{x\},\{y,z\}\} & \mathbb{X}_{\{a_2\}}^\gamma = \{\emptyset\} \\ \mathbb{X}_{\{a_1,a_2\}} = \emptyset & \mathbb{X}_{\{a_1,a_2\}}^\gamma = \emptyset \end{array}$$

Moreover, note that $\text{Pos}(\{a_1\}) = \{z\} = \text{Pos}(\{a_1, a_2\})$ and $\text{Pos}(\{a_2\}) = \emptyset$. Finally, we also have $\text{Red}_\gamma = \{\{a_1\}\}$.

3. Connection of (γ-)decision reducts and (γ-)decision bireducts with definability in a Pawlak approximation space

We start with discussing some relationships between the different notions of reduction. As illustrated in Example 1, it is clear that there is no connection between them for an inconsistent decision system. However, for a consistent decision system, we have the following result:

Theorem 2. *Let $\mathbb{A}$ be a consistent decision system, i.e., $\text{Pos}(C) = U$, and $B \subseteq C$, then*

$$B \in \text{Red} \Leftrightarrow B \in \text{Red}_\gamma \Leftrightarrow \mathbb{X}_B = \{U\} \Leftrightarrow \mathbb{X}_B^\gamma = \{U\}.$$

Proof. Immediately from Definitions 4, 5 and 6. □

Hence, if the set of conditional attributes B is a (γ-)decision reduct, then the only subset of the universe which can operate as a component to obtain a (γ-)decision bireduct is the universe itself. Vice versa, by computing the (γ-) decision bireducts of a consistent decision system, we can easily obtain the set of (γ-)decision reducts. We illustrate this in the following example:

Example 2. Let $\mathbb{A}$ be the decision system with $U = \{x, y, z\}$ and $C = \{a_1, a_2\}$ presented in Table 2. The system $\mathbb{A}$ is consistent. We obtain that $\text{Red} = \text{Red}_\gamma\{\{a_2\}\}$. Moreover, we have the following decision bireducts and γ-decision bireducts:

$$\mathbb{X}_{\emptyset} = \emptyset \qquad \mathbb{X}^{\gamma}_{\emptyset} = \emptyset$$
$$\mathbb{X}_{\{a_1\}} = \{\{x, z\}, \{y, z\}\} \qquad \mathbb{X}^{\gamma}_{\{a_1\}} = \{\{z\}\}$$
$$\mathbb{X}_{\{a_2\}} = \{\{x, y, z\}\} \qquad \mathbb{X}^{\gamma}_{\{a_2\}} = \{\{x, y, z\}\}$$
$$\mathbb{X}_{\{a_1,a_2\}} = \emptyset \qquad \mathbb{X}^{\gamma}_{\{a_1,a_2\}} = \emptyset$$

Next, we want to investigate whether there is a connection between the definable sets in the Pawlak approximation space (U, E_B) and the sets $X \subseteq U$ such that (B, X) is a decision bireduct, respectively γ-decision bireduct. First, we prove that there is a connection between the family of definable sets $\mathcal{B}(U/E_B)$ and the set $\mathbb{X}^{\gamma}_B$.

Theorem 3. *Let $\mathbb{A}$ be a decision system and $B \subseteq C$, then $\mathbb{X}^{\gamma}_B \subseteq \mathcal{B}(U/E_B)$.*

Proof. If $\mathbb{X}^{\gamma}_B = \emptyset$, then $\mathbb{X}^{\gamma}_B \subseteq \mathcal{B}(U/E_B)$, since the Boolean algebra $\mathcal{B}(U/E_B)$ contains the empty set. On the other hand, if $\mathbb{X}^{\gamma}_B \neq \emptyset$, then $\mathbb{X}^{\gamma}_B = \{\mathrm{Pos}(B)\}$ by Theorem 1. By definition of the positive region of B, $\mathrm{Pos}(B)$ is a definable set, as it is the union of a set of equivalence classes based on the indiscernibility relation E_B:

$$\mathrm{Pos}(B) = \bigcup \{[y]_{E_B} \in U/E_B \mid y \in U, \exists x \in U : [y]_{E_B} \subseteq [x]_{E_d}\}.$$

Hence, $\mathbb{X}^{\gamma}_B \subseteq \mathcal{B}(U/E_B)$. □

Note that the inclusion of Theorem 3 is strict:

Example 3. Let $\mathbb{A}$ be the decision system represented in Table 2 and let $B = \{a_2\}$. We obtain that $\mathbb{X}^{\gamma}_B = \{\{x, y, z\}\}$ and that $\mathcal{B}(U/E_B) = \{\emptyset, \{x\}, \{y, z\}, \{x, y, z\}\}$. It is clear that $\mathbb{X}^{\gamma}_B \subsetneq \mathcal{B}(U/E_B)$, as $\{x\}$ is a definable set in (U, E_B), but $(B, \{x\})$ is not a γ-decision bireduct.

For a consistent decision system, we have the following connection between $\mathcal{B}(U/E_B)$ and the set $\mathbb{X}_B$, when B is a decision reduct.

Theorem 4. *Let $\mathbb{A}$ be a consistent decision system and $B \in$ Red, then $\mathbb{X}_B \subseteq \mathcal{B}(U/E_B)$.*

Proof. From Theorem 2 we obtain that $\mathbb{X}_B = \{U\} \subseteq \mathcal{B}(U/E_B)$. □

Unfortunately, we have no general connection between the set of definable sets $\mathcal{B}(U/E_B)$ and the set $\mathbb{X}_B$.

Example 4. Let $\mathbb{A}$ be the decision system represented in Table 2 and let $B = \{a_1\}$. We obtain that $\mathbb{X}_B = \{\{x, z\}, \{y, z\}\}$ and that $\mathcal{B}(U/E_B) = \{\emptyset, \{x, y\}, \{z\}, \{x, y, z\}\}$. On the one hand, we have that $(B, \{x, z\})$ is a decision bireduct, but $\{x, z\}$ is not definable in (U, E_B). On the other hand, the set $\{x, y\}$ is definable in (U, E_B), but it does not form a decision bireduct with B.

4. Connection of (γ-)decision reducts and (γ-)decision bireducts with definability in a covering approximation space

In this section we generalize the concepts of discernibility, decision reducts, bireducts and γ-bireducts to covering approximation spaces. Furthermore, we study which results concerning reduction and definability remain valid in the more general setting.

4.1. Discernibility in a covering approximation space

A crucial point in the definition of a decision reduct is the definition of discernibility of objects by conditional attributes in a given subset $B \subseteq C$. An object $x \in U$ is discernible from $y \in U$ in the Pawlak approximation space (U, E_B) if $[x]_{E_B} \neq [y]_{E_B}$, or if there exists a set K in the partition U/E_B such that $x \in K$ and $y \notin K$. This latter statement can be easily generalized for a covering approximation space $(U, \mathbb{C}_B)$:

Definition 7. Let $\mathbb{A}$ be a decision system and $B \subseteq C$ a set of conditional attributes. We say that the object $x \in U$ is *discernible from* the object $y \in U$ in the covering approximation space $(U, \mathbb{C}_B)$ if there exists a set $K \in \mathbb{C}_B$ such that $x \in K$ and $y \notin K$.

Note that in a Pawlak approximation space the discernibility relation is an equivalence relation since U/E_B is a partition. In particular, the discernibility relation is symmetric: x is discernible from y in (U, E_B) if and only if y is discernible from x in (U, E_B). This is no longer necessarily the case in a covering approximation space $(U, \mathbb{C}_B)$. For example, assume $\mathbb{C}_B = \{\{x\}, \{x, y\}\}$, then x is discernible from y, but y is not discernible from x. On the other hand, in [1], Benítez-Caballero et al. considered discernibility based on a tolerance relation R_B, in which the corresponding discernibility relation is reflexive and symmetric (but not necessarily transitive).

In the light of this difference, consistency needs to be redefined in the scope of a covering approximation space $(U, \mathbb{C}_C)$: $\mathbb{A}$ is consistent if for each pair $x, y \in U$ with different decision values the object x is discernible from the object y in $(U, \mathbb{C}_C)$ and if y is discernible from the object x in $(U, \mathbb{C}_C)$.

It is possible to give different characterizations of discernibility in a covering approximation space. For this purpose, we recall the following two notions related to a covering $\mathbb{C}$.

Definition 8. [3] Let $\mathbb{C}$ be a covering of U. The *neighborhood of an object* $x \in U$ is defined as

$$N(x) = \bigcap\{K \in \mathbb{C} \mid x \in K\}.$$

Moreover, we obtain another covering, called *the covering induced by* $\mathbb{C}$, which is defined as

$$\mathrm{Cov}(\mathbb{C}) = \{N(x) \mid x \in U\}.$$

Remark 1. Note that $N(x)$ is sometimes denoted by C_x [3], $Neighbor(x)$ [27] or $N_1^{\mathbb{C}}(x)$ [16]. Moreover, $\mathrm{Cov}(\mathbb{C})$ is sometimes denoted by $\mathbb{C}_3$ [25].

The neighborhood $N(x)$ of an object x has the following properties: $x \in N(x)$ and if $y \in N(x)$, then $N(y) \subseteq N(x)$. In addition, note that for a partition U/E, we have $N(x) = [x]_E$ and $\mathrm{Cov}(U/E) = U/E$. For a set of conditional attributes $B \subseteq C$, we denote the neighborhood of x by $N_B(x)$ to emphasize the relation with the covering approximation space $(U, \mathbb{C}_B)$.

We have the following property which allows us to use different characterizations for discernibility in a covering approximation space.

Proposition 1. *Let $\mathbb{C}$ be a covering of U and $x, y \in U$, then the following statements are equivalent:*

(1) *There is a set $K \in \mathbb{C}$ such that $x \in K$ and $y \notin K$;*
(2) *$y \notin N(x)$;*
(3) *There is a set $K \in \mathrm{Cov}(\mathbb{C})$ such that $x \in K$ and $y \notin K$.*

Proof.
- (1) ⇔ (2): For every set $K \in \mathbb{C}$ with $x \in K$ it holds that $N(x) \subseteq K$, hence if $K \in \mathbb{C}$ such that $x \in K$ and $y \notin K$, then $y \notin N(x)$. On the other hand, if $y \notin N(x)$, then by definition of $N(x)$ there exists a set $K \in \mathbb{C}$ with $x \in K$ and $y \notin K$.
- (2) ⇔ (3): Since $N(x) \in \mathrm{Cov}(\mathbb{C})$ and $x \in N(x)$, if $y \notin N(x)$, then there is a set $K \in \mathrm{Cov}(\mathbb{C})$ such that $x \in K$ and $y \notin K$. On the other hand, assume $K \in \mathrm{Cov}(\mathbb{C})$ with $x \in K$ and $y \notin K$. As $x \in K$, we have $N(x) \subseteq K$. As $y \notin K$, we have $y \notin N(x)$. □

Next, we discuss the relationship between the union-closure of a covering and the union-closure of its induced covering.

Proposition 2. *Let $(U, \mathbb{C})$ be a covering approximation space, then*

1. $\cup^*(\mathbb{C}) \subseteq \cup^*(\mathrm{Cov}(\mathbb{C}))$,
2. $\cup^*(\mathrm{Cov}(\mathbb{C})) \subseteq \cup^*(\mathbb{C}) \Leftrightarrow \mathrm{Cov}(\mathbb{C}) \subseteq \mathbb{C}$.

Proof.
1. Let $K \in \mathbb{C}$, we claim that $K = \bigcup_{x \in K} N(x)$. As $x \in N(x)$ for each $x \in U$, it holds that $K \subseteq \bigcup_{x \in K} N(x)$. On the other hand, for every $x \in K$ it holds that $N(x) \subseteq K$, thus, $\bigcup_{x \in K} N(x) \subseteq K$.
   Let $X \in \cup^*(\mathbb{C})$, then there exists a subset $F \subseteq \mathbb{C}$ such that $X = \bigcup_{K \in F} K$, thus $X = \bigcup_{K \in F} \bigcup_{x \in K} N(x)$. Therefore, we can write X as follows: $X = \bigcup_{K' \in F'} K'$ with $F' = \{N(x) \mid x \in K, K \in F\} \subseteq \mathrm{Cov}(\mathbb{C})$. We conclude that $X \in \cup^*(\mathrm{Cov}(\mathbb{C}))$.
2. If $\mathrm{Cov}(\mathbb{C}) \subseteq \mathbb{C}$, then $\cup^*(\mathrm{Cov}(\mathbb{C})) \subseteq \cup^*(\mathbb{C})$ holds by the definition of union-closure. On the other hand, assume $\cup^*(\mathrm{Cov}(\mathbb{C})) \subseteq \cup^*(\mathbb{C})$ and take $N(x) \in \mathrm{Cov}(\mathbb{C})$ for $x \in U$. Since $N(x) \in \cup^*(\mathrm{Cov}(\mathbb{C}))$, $N(x) \in \cup^*(\mathbb{C})$, thus there exists a subset $F \subseteq \mathbb{C}$ such that $N(x) = \bigcup_{K \in F} K$. Since $x \in N(x)$, there is a set $K \in F$ such that $x \in K$. By definition of the union, we have that $K \subseteq N(x)$ and by definition of the neighborhood of x we have that $N(x) \subseteq K$. Hence, $N(x) = K$ and we conclude that $N(x) \in \mathbb{C}$.

This concludes the proof. □

Given the definition of discernibility in a covering approximation space and different characterizations of this concept available, we can now introduce the concepts of decision reduct, γ-decision reduct, decision bireduct and γ-decision bireduct in a covering approximation space.

4.2. Definitions of (γ-)decision reducts and (γ-)decision bireducts in a covering approximation space

We start by introducing decision reducts in a covering approximation space.

Definition 9. Let $\mathbb{A}$ be a consistent decision system. A set of conditional attributes $B \subseteq C$ is a *decision reduct* for the decision system $\mathbb{A}$ if and only if for each pair $x, y \in U$ satisfying the inequality $I_d(x) \neq I_d(y)$, x is discerned from y in $(U, \mathbb{C}_B)$, and if B is irreducible with respect to this property, i.e., there is no smaller subset $B' \subseteq B$ which satisfies this property.
We denote cRed for the set of decision reducts obtained using covering approximation spaces.

Note that $B = \emptyset$ is only a decision reduct when $U/d = \{U\}$, similarly as in a Pawlak approximation space.
Next, we introduce γ-decision reducts in covering approximation spaces. For this, we first need to define an appropriate notion of positive region, which in turn depends on the choice of lower approximation. As we have mentioned before, in this paper we focus on the tight lower approximation. In particular, for reasons that will become clear below, we define the positive region of $B \subseteq C$ using $\underline{\text{apr}}'_{\text{Cov}(\mathbb{C}_B)}$. Note also that we still assume that the decision attribute d is associated with an equivalence relation E_d:

$$\text{cPos}(B) = \bigcup_{[x]_{E_d} \in U/E_d} \underline{\text{apr}}'_{\text{Cov}(\mathbb{C}_B)}([x]_{E_d}) = \bigcup_{[x]_{E_d} \in U/E_d} \bigcup \{K \in \text{Cov}(\mathbb{C}_B) \mid K \subseteq [x]_{E_d}\} = \{x \in U \mid N_B(x) \subseteq [x]_{E_d}\}.$$

We first verify that consistency of a decision system still implies that the positive region equals U. Indeed, let $x \in U$ and suppose $I_d(x) \neq I_d(y)$. Since x is discernible from y, there exists a $K \in \mathbb{C}_C$ such that $x \in K$ and $y \notin K$, hence $y \notin N_B(x)$. In other words, $N_B(x) \subseteq [x]_{E_d}$ and $x \in \text{cPos}(C)$.

Definition 10. Let $\mathbb{A}$ be a decision system. A set of conditional attributes $B \subseteq C$ is a *γ-decision reduct* for the decision system $\mathbb{A}$ if and only if $\text{cPos}(B) = \text{cPos}(C)$, and if B is irreducible with respect to this property, i.e., there is no smaller subset $B' \subseteq B$ which satisfies this property. We denote cRed_γ for the set of decision reducts obtained using covering approximation spaces.

Analogously as in Pawlak's rough set model, we can maintain the property that for a consistent decision system, the two notions of reduct coincide. This is proven formally in the following theorem:

Theorem 5. *Let $\mathbb{A}$ be a consistent decision system and $B \subseteq C$, then $B \in$ cRed if and only if $B \in \text{cRed}_\gamma$.*

Proof. We will prove that $\text{cPos}(B) = \text{cPos}(C)$ if and only if for each pair $x, y \in U$ satisfying the inequality $I_d(x) \neq I_d(y)$, x is also discerned from y in $(U, \mathbb{C}_B)$.

- Suppose first that $B \in$ cRed, and consider $x \in \text{cPos}(C)$. For $y \in U$ such that $I_d(x) \neq I_d(y)$, it holds that there exists a $K \in \mathbb{C}_B$ such that $x \in K$ and $y \notin K$. By Proposition 1, this also means $y \notin N_B(x)$. From this, we infer $N_B(x) \subseteq [x]_{E_d}$, in other words $x \in \text{cPos}(B)$. Together with $\text{cPos}(B) \subseteq \text{cPos}(C) = U$, this implies that $\text{cPos}(B) = \text{cPos}(C)$.
- Suppose $\text{cPos}(B) = \text{cPos}(C) = U$. Consider $x, y \in U$ such that $I_d(x) \neq I_d(y)$. Since x is discerned from y in $(U, \mathbb{C}_C)$, there exists $K \in \mathbb{C}_C$ such that $x \in K$ and $y \notin K$. Assume now that x is not discerned from y in $(U, \mathbb{C}_B)$, in other words for all $K \in \mathbb{C}_B$ such that $x \in K$, also $y \in K$. From this follows that $y \in N_B(x)$, and thus also $N_B(x) \nsubseteq [x]_{E_d}$. Hence, $x \notin \text{cPos}(B)$, a contradiction. □

The following example shows that the use of the induced covering $\text{Cov}(\mathbb{C}_B)$, rather than $\mathbb{C}_B$ itself, is necessary in the definition of $\text{cPos}(B)$ to maintain the equivalence of Theorem 5.

Example 5. Let $\mathbb{A}$ be the consistent decision system with $U = \{x, y, z\}$ presented in Table 3 and $C = \{a_1, a_2\}$. Define $\mathbb{C}_{\{a_1\}} = \{R_1(u) \mid u \in U\}$, with $R_1(u) = \{v \in U \mid v \neq u \wedge a_1(v) = a_1(u)\}$, and $\mathbb{C}_{\{a_2\}} = \{R_2(u) \mid u \in U\}$, with $R_2(u) = \{v \in U \mid a_2(v) = a_2(u)\}$. Then $\mathbb{C}_{\{a_1\}} = \{\{y, z\}, \{x, z\}, \{x, y\}\}$ and $\mathbb{C}_{\{a_2\}} = \{\{x\}, \{y\}, \{z\}\}$. Clearly, $\text{cRed} = \{\{a_1\}, \{a_2\}\}$, and

$$N_{\{a_1\}}(x) = N_{\{a_2\}}(x) = \{x\}, N_{\{a_1\}}(y) = N_{\{a_2\}}(y) = \{y\}, N_{\{a_1\}}(z) = N_{\{a_2\}}(z) = \{z\}.$$

From this, we find $\text{cPos}(\{a_1\}) = \text{cPos}(\{a_2\}) = \text{cPos}(C) = \{x, y, z\}$, as expected. On the other hand,

$$\bigcup_{[u]_{E_d} \in U/E_d} \underline{\text{apr}}'_{\mathbb{C}_{\{a_1\}}}([u]_{E_d}) = \underline{\text{apr}}'_{\mathbb{C}_{\{a_1\}}}(\{x\}) \cup \underline{\text{apr}}'_{\mathbb{C}_{\{a_1\}}}(\{y\}) \cup \underline{\text{apr}}'_{\mathbb{C}_{\{a_1\}}}(\{z\}) = \emptyset,$$

Table 3
Decision system $\mathbb{A}$ from Example 5.

	a_1	a_2	d
x	1	0	0
y	1	0.5	1
z	1	1	2

while

$$\bigcup_{[u]_{E_d} \in U/E_d} \underline{\text{apr}}'_{\mathbb{C}_C}([u]_{E_d}) = \{x, y, z\}.$$

Next, we define decision bireducts in covering approximation spaces.

Definition 11. Let $\mathbb{A}$ be a decision system. A pair (B, X) with $B \subseteq C$ and $X \subseteq U$ is a *decision bireduct* if and only if for each pair $x, y \in X$ satisfying inequality $I_d(x) \neq I_d(y)$ the object x is discerned from the object y in the covering approximation space $(U, \mathbb{C}_B)$, y is discerned from x in $(U, \mathbb{C}_B)$ and the following properties hold:

1. There is no $B' \subsetneq B$ such that for each pair $x, y \in X$ satisfying inequality $I_d(x) \neq I_d(y)$ the object x is discerned from the object y in the covering approximation space $(U, \mathbb{C}_{B'})$ and y is discerned from x in $(U, \mathbb{C}_{B'})$.
2. There is no $X' \supsetneq X$ such that for each pair $x, y \in X'$ satisfying inequality $I_d(x) \neq I_d(y)$ the object x is discerned from the object y in the covering approximation space $(U, \mathbb{C}_B)$ and y is discerned from x in $(U, \mathbb{C}_B)$.

We denote $\text{c}\mathbb{X}_B$ for the set $\{X \subseteq U \mid (B, X) \text{ decision bireduct}\}$ when covering approximation spaces are used.

Finally, γ-decision bireducts in covering approximation spaces are defined.

Definition 12. Let $\mathbb{A}$ be a decision system. A pair (B, X) with $B \subseteq C$ and $X \subseteq U$ is a *γ-decision bireduct* if and only if for each pair $x \in X, y \in U$ satisfying inequality $I_d(x) \neq I_d(y)$ the object x is discerned from the object y in the covering approximation space $(U, \mathbb{C}_B)$ and the following properties hold:

1. There is no $B' \subsetneq B$ such that for each pair $x \in X, y \in U$ satisfying inequality $I_d(x) \neq I_d(y)$ the object x is discerned from the object y in the covering approximation space $(U, \mathbb{C}_{B'})$.
2. There is no $X' \supsetneq X$ such that for each pair $x \in X', y \in U$ satisfying inequality $I_d(x) \neq I_d(y)$ the object x is discerned from the object y in the covering approximation space $(U, \mathbb{C}_B)$.

We denote $\text{c}\mathbb{X}_B^\gamma$ for the set $\{X \subseteq U \mid (B, X)\ \gamma\text{-decision bireduct}\}$ when covering approximation spaces are used.

Note that for a decision bireduct (B, X) it holds that

$$\forall x, y \in X\colon I_d(x) \neq I_d(y) \Rightarrow y \notin N_B(x) \text{ and } x \notin N_B(y),$$

while for a γ-decision bireduct (B, X) we have that

$$\forall x \in X, \forall y \in U\colon I_d(x) \neq I_d(y) \Rightarrow y \notin N_B(x).$$

Moreover, $X = \emptyset$ can never form a decision bireduct with $B \subseteq C$, since we can always extend X with an object $x \in U$. However, it is possible that $(B, \emptyset)$ is a γ-decision bireduct. In addition, analogously as in a Pawlak approximation spaces, it is possible that the set $B \subseteq C$ forms a decision bireduct with different sets of objects and that the set $X \subseteq U$ forms a decision bireduct with different sets of conditional attributes. On the other hand, given $B \subseteq C$, there is at most one subset $X \subseteq U$ such that (B, X) is a γ-decision bireduct, namely $X = \text{cPos}(B)$. The latter observation is summed up in the following analogon of Theorem 1.

Theorem 6. *Let $\mathbb{A}$ be a decision system and $B \subseteq C$. If $\text{c}\mathbb{X}_B^\gamma \neq \emptyset$, then $\text{c}\mathbb{X}_B^\gamma = \{\text{cPos}(B)\}$.*

Proof. We first show that $|\text{c}\mathbb{X}_B^\gamma| \leq 1$. If $\text{c}\mathbb{X}_B^\gamma = \emptyset$, then $|\text{c}\mathbb{X}_B^\gamma| = 0$. Assume that $\text{c}\mathbb{X}_B^\gamma = \{X, Y\}$ with $X \neq Y$. Without loss of generality, take $x \in X \setminus Y$. Since (B, X) is a γ-decision bireduct and $x \in X$, we have that $\forall z \in U\colon I_d(x) \neq I_d(z) \Rightarrow z \notin N_B(x)$. On the other hand, since (B, Y) is a γ-decision bireduct and $x \notin Y$, it holds that $\exists z \in U\colon I_d(x) \neq I_d(z) \wedge z \in N_B(x)$. This is a contradiction. Hence, we conclude that $\text{c}\mathbb{X}_B^\gamma$ contains at most one set of objects.

Now assume that $\text{c}\mathbb{X}_B^\gamma = \{X\}$. We prove that $x \in X$ if and only if $N_B(x) \subseteq [x]_{E_d}$:

- Assume $x \in X$, then $N_B(x) \subseteq [x]_{E_d}$ since (B, X) is a γ-decision bireduct.
- Assume $x \notin X$, then there exists an object $y \in U$ such that $I_d(x) \neq I_d(y)$ and $y \in N_B(x)$. Hence, $N_B(x) \nsubseteq [x]_{E_d}$.

We conclude that X is given by $\{x \in U \mid N_B(x) \subseteq [x]_{E_d}\} = \text{cPos}(B)$. □

4.3. Results concerning reduction and definability in a covering approximation space

In this section, we study whether the results from Section 3 concerning (γ-)decision bireducts are still valid when covering approximation spaces are considered. First, we study the relation between decision reducts, decision bireducts and γ-decision bireducts for a consistent decision system $\mathbb{A}$.

First, we show that Theorem 2 remains valid:

Theorem 7. *Let $\mathbb{A}$ be a consistent decision system and $B \subseteq C$, then*

$$B \in \text{cRed} \Leftrightarrow \text{c}\mathbb{X}_B = \{U\} \Leftrightarrow \text{c}\mathbb{X}_B^\gamma = \{U\}.$$

Proof. Immediately from Definitions 9, 11 and 12. □

In the following, we study the connection between $\text{c}\mathbb{X}_B^\gamma$ and the set of definable sets of $(U, \mathbb{C}_B)$ given by $\cup^*(\mathbb{C}_B)$, for $B \subseteq C$. In Pawlak's setting, $\mathbb{X}_B^\gamma \subseteq \mathcal{B}(U/E_B)$. However, we do not have $\text{c}\mathbb{X}_B^\gamma \subseteq \cup^*(\mathbb{C}_B)$ as illustrated in the next example.

Example 6. Let $\mathbb{A}$ be a decision system with $U = \{v, w, x, y, z\}$, $U/E_d = \{\{v, w, x\}, \{w, y\}\}$ and let $a \in C$ be a conditional attribute such that $I_a(v) = I_a(w) = 0$, $I_a(x) = I_a(y) = 1$ and $I_a(z) = 0.5$. We construct the covering $\mathbb{C}_{\{a\}}$ in the following way: $\mathbb{C}_{\{a\}} = \{R(x) \mid x \in U\}$ with $y \in R(x)$ if and only if $|a(x) - a(y)| \leq 0.5$. It holds that:

$$\mathbb{C}_{\{a\}} = \{\{v, w, z\}, \{x, y, z\}, \{v, w, x, y, z\}\}.$$

Hence, $\cup^*(\mathbb{C}_{\{a\}}) = \{\emptyset, \{v, w, z\}, \{x, y, z\}, \{v, w, x, y, z\}\}$. For the attribute a we have $\text{c}\mathbb{X}_{\{a\}}^\gamma = \{\{z\}\}$. We conclude that $\text{c}\mathbb{X}_{\{a\}}^\gamma$ is not a subset of $\cup^*(\mathbb{C}_{\{a\}})$.

Note that for a partition U/E it holds that $\text{Cov}(U/E) = U/E$, hence, Theorem 3 also states that $\mathbb{X}_B^\gamma \subseteq \mathcal{B}(\text{Cov}(U/E))$. This result can be generalized for covering approximation spaces. Indeed, if we consider instead the family of definable sets in the union-closure $\cup^*(\text{Cov}(\mathbb{C}_B))$, we can obtain the following extension of Theorem 3:

Theorem 8. *Let $\mathbb{A}$ be a decision system and $B \subseteq C$, then $\text{c}\mathbb{X}_B^\gamma \subseteq \cup^*(\text{Cov}(\mathbb{C}_B))$.*

Proof. If $\text{c}\mathbb{X}_B^\gamma = \emptyset$, then the inclusion holds trivially. On the other hand, let $X \in \text{c}\mathbb{X}_B^\gamma$ with $X = \{x \in U \mid N_B(x) \subseteq [x]_{E_d}\}$. We claim that $X = \bigcup_{x \in X} N_B(x)$:

- Let $x \in X$. Since $x \in N_B(x)$ it holds that $X \subseteq \bigcup_{x \in X} N_B(x)$.
- Let $y \in \bigcup_{x \in X} N_B(x)$, then there is an object $x \in X$ such that $y \in N_B(x)$, hence, $N_B(y) \subseteq N_B(x)$. Since $x \in X$ we have $N_B(x) \subseteq [x]_{E_d}$ and since $y \in N_B(x)$, we have $[x]_{E_d} = [y]_{E_d}$. Thus, $N_B(y) \subseteq [y]_{E_d}$. Therefore, $y \in X$.

As $\text{Cov}(\mathbb{C}_B) = \{N_B(x) \mid x \in U\}$, it holds that $X \in \cup^*(\text{Cov}(\mathbb{C}_B))$. □

Example 7. Recall the decision system of Example 6. We have that $\text{Cov}(\mathbb{C}_{\{a\}}) = \{\{v, w, z\}, \{x, y, z\}, \{z\}\}$ and

$$\cup^*(\text{Cov}(\mathbb{C}_{\{a\}})) = \{\emptyset, \{v, w, z\}, \{x, y, z\}, \{z\}, \{v, w, x, y, z\}\}.$$

The set of definable sets $\cup^*(\mathbb{C}_B)$ does not contain the set $\text{c}\mathbb{X}_B^\gamma$ since the sets of $\mathbb{C}_B$ are 'too big'. By considering $\cup^*(\text{Cov}(\mathbb{C}_B))$ we allow sets which have been built by intersection (Cov) and/or union ($\cup^*$).

5. Worked example

In this section, we will illustrate the concepts studied in this paper in a simple decision system, taken as a sample from the Pima Indians Diabetes data set located at the UCI Machine Learning repository,[1] and also considered in [4].

[1] Available at http://www.ics.uci.edu/~mlearn/MLRepository.html.

Table 4
Decision system $\mathbb{A}$ from Section 5, and its normalized version.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	d
x_1	1	101	50	15	36	24.2	0.526	26	0
x_2	8	176	90	34	300	33.7	0.467	58	1
x_3	7	150	66	42	342	34.7	0.718	42	0
x_4	7	187	68	39	304	37.7	0.254	41	1
x_5	0	100	88	60	110	46.8	0.962	31	0
x_6	0	105	64	41	142	41.5	0.173	22	0
x_7	1	95	66	13	38	19.6	0.334	25	0

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	d
x_1	0.125	0.065	0.000	0.043	0.000	0.169	0.447	0.111	0
x_2	1.000	0.880	1.000	0.447	0.863	0.518	0.373	1.000	1
x_3	0.875	0.598	0.400	0.617	1.000	0.555	0.691	0.556	0
x_4	0.875	1.000	0.450	0.553	0.876	0.665	0.103	0.528	1
x_5	0.000	0.054	0.950	1.000	0.242	1.000	1.000	0.250	0
x_6	0.000	0.109	0.350	0.596	0.346	0.805	0.000	0.000	0
x_7	0.125	0.000	0.400	0.000	0.007	0.000	0.204	0.083	0

The original decision system is shown in the upper part of Table 4. It contains seven objects ($U = \{x_1, \ldots, x_7\}$) and eight conditional attributes that are all quantitive ($C = \{a_1, \ldots, a_8\}$). We have one qualitative decision attribute d with two possible values, indicating the absence ($d = 0$) or presence of diabetes ($d = 1$). In order to make the construction of coverings for individual attributes uniform, we first divide all conditional attributes' values by their range. The normalized decision system is shown in the lower part of Table 4.

We define the coverings $\mathbb{C}_{\{a_i\}}$, $i = 1, \ldots, 7$, as follows: $\mathbb{C}_{\{a_i\}} = \{R_i(x) \mid x \in U\}$ with $y \in R_i(x)$ if and only if $|a_i(x) - a_i(y)| \leq 0.2$. We thus find:

$$\mathbb{C}_{\{a_1\}} = \{\{x_1, x_5, x_6, x_7\}, \{x_2, x_3, x_4\}\}$$
$$\mathbb{C}_{\{a_2\}} = \{\{x_1, x_5, x_6, x_7\}, \{x_2, x_4\}, \{x_3\}\}$$
$$\mathbb{C}_{\{a_3\}} = \{\{x_1\}, \{x_2, x_5\}, \{x_3, x_4, x_6, x_7\}\}$$
$$\mathbb{C}_{\{a_4\}} = \{\{x_1, x_7\}, \{x_2, x_3, x_4, x_6\}, \{x_5\}\}$$
$$\mathbb{C}_{\{a_5\}} = \{\{x_1, x_7\}, \{x_2, x_3, x_4\}, \{x_5, x_6\}\}$$
$$\mathbb{C}_{\{a_6\}} = \{\{x_1, x_7\}, \{x_2, x_3, x_4\}, \{x_4, x_6\}, \{x_5, x_6\}\}$$
$$\mathbb{C}_{\{a_7\}} = \{\{x_1, x_2\}, \{x_2, x_4, x_7\}, \{x_2, x_7\}, \{x_3\}, \{x_4, x_6\}, \{x_4, x_6, x_7\}, \{x_5\}\}$$
$$\mathbb{C}_{\{a_8\}} = \{\{x_1, x_5, x_6, x_7\}, \{x_1, x_5, x_7\}, \{x_1, x_6, x_7\}, \{x_2\}, \{x_3, x_4\}\}$$

Note that the first five coverings are partitions, while the last three ones are not. For these last three, we also list the corresponding induced coverings $\mathrm{Cov}(\mathbb{C}_{\{a_i\}})$, which will be needed in the computation of the positive region:

$$\mathrm{Cov}(\mathbb{C}_{\{a_6\}}) = \{\{x_1, x_7\}, \{x_2, x_3, x_4\}, \{x_4\}, \{x_4, x_6\}, \{x_5, x_6\}, \{x_6\}\}$$
$$\mathrm{Cov}(\mathbb{C}_{\{a_7\}}) = \{\{x_1, x_2\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_4, x_6\}, \{x_5\}, \{x_7\}\}$$
$$\mathrm{Cov}(\mathbb{C}_{\{a_8\}}) = \{\{x_1, x_5, x_7\}, \{x_1, x_6, x_7\}, \{x_1, x_7\}, \{x_2\}, \{x_3, x_4\}\}$$

For general $B \subseteq C$, the coverings $\mathbb{C}_B$ are constructed by taking non-empty intersections of sets in $\mathbb{C}_{\{a\}}$ for all $a \in B$. For example,

$$\mathbb{C}_{\emptyset} = \{\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}\}$$
$$\mathbb{C}_{\{a_1, a_2\}} = \{\{x_1, x_5, x_6, x_7\}, \{x_2, x_4\}, \{x_3\}\}$$
$$\mathbb{C}_{\{a_1, a_3\}} = \{\{x_1\}, \{x_2\}, \{x_3, x_4\}, \{x_5\}, \{x_6, x_7\}\}$$
$$\mathbb{C}_C = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_6\}, \{x_7\}\}$$

The last equality also makes it clear that the decision system is consistent, as all objects (and, a fortiori, all objects in different classes) can be discerned by the full set of conditional attributes. As a consequence, the sets of decision reducts, cRed, and the set of γ-decision reducts, $cRed_\gamma$, are equal in this case. We find them by computing the positive region of subsets of attributes, starting with singleton subsets and consecutively adding attributes to the non-reduct ones, taking into account the minimality condition for decision reducts. For example, $\mathrm{cPOS}(\{a_1\}) = \{x_1, x_5, x_6, x_7\}$ and $\mathrm{cPOS}(\{a_2\}) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, ensuring that $\{a_2\}$ is a decision reduct and $\{a_1\}$ is not. In this way, we obtain the following set of (γ-) decision reducts:

$$\text{cRed} = \text{cRed}_\gamma = \{\{a_2\}, \{a_7\}, \{a_3, a_6\}, \{a_6, a_8\}, \{a_1, a_3, a_4\}\}$$

In order to find decision bireducts and γ-decision bireducts, we operate in a similar way as for decision reducts, starting with the empty set and consecutively adding attributes, checking for each subset $B \subseteq C$ whether there exist maximal subsets X of U such that discernibility is preserved (between elements in X for decision reducts, and between elements in X and all elements in U for γ-decision reducts).

For γ-decision reducts, the task is simplified by Theorem 6; indeed, given $B \subseteq C$, $c\mathbb{X}_B^\gamma = \{\text{cPOS}(B)\}$, provided there does not exist a $B' \subsetneq B$ such that $\text{cPOS}(B) = \text{cPOS}(B')$.

For example, both $c\mathbb{X}_\emptyset$ and $c\mathbb{X}_\emptyset^\gamma$ are empty, while $c\mathbb{X}_{\{a_1\}} = \{\{x_1, x_2, x_4, x_5, x_6, x_7\}, \{x_1, x_3, x_5, x_6, x_7\}\}$. Indeed, when considering only the attribute a_1, every X in $c\mathbb{X}_{\{a_1\}}$ contains at least $\{x_1, x_5, x_6, x_7\}$, i.e., the patch $K \in \mathbb{C}_{\{a_1\}}$ which only contains elements from the non-diabetes decision class. We can expand this patch either with x_3, or with $\{x_2, x_4\}$ without creating indiscernibilities between elements of opposite classes. On the other hand, $c\mathbb{X}_{\{a_1\}}^\gamma = \{\{x_1, x_5, x_6, x_7\}$, since those are the elements belonging to the positive region of this attribute.

Finally, all decision bireducts and γ-decision bireducts are given as:

Decision bireducts	γ-decision bireducts
$c\mathbb{X}_{\{a_1\}} = \{\{x_1, x_2, x_4, x_5, x_6, x_7\}, \{x_1, x_3, x_5, x_6, x_7\}\}$	$c\mathbb{X}_{\{a_1\}}^\gamma = \{\{x_1, x_5, x_6, x_7\}\}$
$c\mathbb{X}_{\{a_2\}} = \{U\}$	$c\mathbb{X}_{\{a_2\}}^\gamma = \{U\}$
$c\mathbb{X}_{\{a_3\}} = \{\{x_1, x_2, x_3, x_6, x_7\}, \{x_1, x_2, x_4\}, \{x_1, x_4, x_5\}, \{x_1, x_3, x_3, x_5, x_6, x_7\}\}$	$c\mathbb{X}_{\{a_3\}}^\gamma = \{\{x_1, x_5, x_7\}\}$
$c\mathbb{X}_{\{a_4\}} = \{\{x_1, x_2, x_4, x_5, x_7\}, \{x_1, x_3, x_5, x_6, x_7\}\}$	$c\mathbb{X}_{\{a_4\}}^\gamma = \{\{x_1, x_5, x_7\}\}$
$c\mathbb{X}_{\{a_5\}} = \{\{x_1, x_2, x_4, x_5, x_6, x_7\}, \{x_1, x_3, x_5, x_6, x_7\}\}$	$c\mathbb{X}_{\{a_5\}}^\gamma = \{\{x_1, x_5, x_6, x_7\}\}$
$c\mathbb{X}_{\{a_6\}} = \{\{x_1, x_2, x_4, x_5, x_6, x_7\}, \{x_1, x_3, x_4, x_5, x_6, x_7\}\}$	$c\mathbb{X}_{\{a_6\}}^\gamma = \{\{x_1, x_4, x_5, x_6, x_7\}\}$
$c\mathbb{X}_{\{a_7\}} = \{U\}$	$c\mathbb{X}_{\{a_7\}}^\gamma = \{U\}$
$c\mathbb{X}_{\{a_8\}} = \{\{x_1, x_2, x_4, x_5, x_6, x_7\}, \{x_1, x_2, x_3, x_5, x_6, x_7\}\}$	$c\mathbb{X}_{\{a_8\}}^\gamma = \{\{x_1, x_2, x_5, x_6, x_7\}\}$
$c\mathbb{X}_{\{a_1,a_3\}} = \{\{x_1, x_2, x_3, x_5, x_6, x_7\}\}$	$c\mathbb{X}_{\{a_1,a_3\}}^\gamma = \{\{x_1, x_2, x_5, x_6, x_7\}\}$
	$c\mathbb{X}_{\{a_3,a_4\}}^\gamma = \{\{x_1, x_2, x_5, x_7\}\}$
$c\mathbb{X}_{\{a_3,a_5\}} = \{\{x_1, x_2, x_3, x_5, x_6, x_7\}\}$	$c\mathbb{X}_{\{a_3,a_5\}}^\gamma = \{\{x_1, x_2, x_5, x_6, x_7\}\}$
$c\mathbb{X}_{\{a_3,a_6\}} = \{U\}$	$c\mathbb{X}_{\{a_3,a_6\}}^\gamma = \{U\}$
$c\mathbb{X}_{\{a_6,a_8\}} = \{U\}$	$c\mathbb{X}_{\{a_6,a_8\}}^\gamma = \{U\}$
$c\mathbb{X}_{\{a_1,a_3,a_4\}} = \{U\}$	$c\mathbb{X}_{\{a_1,a_3,a_4\}}^\gamma = \{U\}$

with the understanding that the remaining $c\mathbb{X}_B$ and $c\mathbb{X}_B^\gamma$ are empty.

6. Conclusion and future work

In this paper, we have combined different reduction techniques of rough sets with a semantical approach of Pawlak's rough sets. Moreover, we have extended the notion of discernibility of objects to covering approximation spaces, leading to the introduction of decision reducts, decision bireducts and γ-decision bireducts in covering approximation spaces. Both in the Pawlak approximation space (U, E_B) and the covering approximation space $(U, \mathbb{C}_B)$ there is at most one γ-decision bireduct (B, X), in which the set of object X equals the B-positive region $\text{Pos}(B)$, resp. $\text{cPos}(B)$. Moreover, the set X is no longer a definable set related with $(U, \mathbb{C}_B)$, but is definable for the induced covering approximation space $(U, \text{Cov}(\mathbb{C}_B))$.

Future work directives include the study of reduction in particular covering approximation spaces. For example, when dominance relations are used to construct the covering [7,8] or when covering approximation spaces are related to incomplete decision tables [9–11]. On the other hand, computationally efficient procedures for finding all (γ-) decision reducts, similar to the ones proposed for Pawlak's rough set model in [20], need to be devised. Finally, we will also investigate the connection of reduction in fuzzy covering approximation spaces with δ-information reducts and bireducts introduced by Benítez et al. [2].

Conflict of interest statement

No conflict of interest.

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