On the Structure and Interpretation of an Intuitionistic Fuzzy Expert System

Chris Cornelis
Fuzziness and Uncertainty Modelling Research Unit
Department of Mathematics and Computer Science
Ghent University, Krijgslaan 281 (S9), B-9000 Gent, Belgium
chris.cornelis@rug.ac.be

Etienne Kerre
Fuzziness and Uncertainty Modelling Research Unit
Department of Mathematics and Computer Science
Ghent University, Krijgslaan 281 (S9), B-9000 Gent, Belgium
etienne.kerre@rug.ac.be

Abstract

This paper presents a blueprint of an intuitionistic fuzzy expert system (IFES). As such, the paper is a synthesis of our earlier work [3, 4, 5, 6] on various knowledge representation and manipulation aspects of imprecision–tolerant rule–based systems.

Keywords: fuzziness and uncertainty modelling, intuitionistic fuzzy sets, expert systems

1 Introduction and Preliminaries

In the past, fuzzy rule–based systems have convinced us of their high potential to solve imprecise but otherwise (for humans) straightforward tasks. However ample their success, their applicability, in general, remains limited to a class of very simple control jobs, like controlling fluid levels in a reactor, automatical lens focussing in cameras, ...and we are still a very long way from a full–scale artificial reasoning unit. To lift this restriction, in our opinion at least the following two objectives should be met: first, we have to acknowledge that imprecision is a multi–faceted notion, distinguishing amongst others between the concept of *fuzziness* (describing the "soft" or gradual transition from satisfying a predicate to not satisfying it) and that of *uncertainty* (describing the extent to which we are confident of information), which we cannot expect to be adequately represented by a fuzzy set alone; thus we are urged to consider a higher–order extension of the latter. Secondly, we cast doubts about the commonplace practise of using interpolation techniques at the ground level of a fuzzy rule–based system (in accordance with the tradition initiated by Mamdani in [11]) to create an illusion of "calculating with words" perceived at user level, as it harms their intuitive feeding ground: namely, that they extend the Modus Ponens (MP) inference rule to an environment with more than two truth–values. As will be argued in this paper, the trend towards simplification leaves little opportunity to deal with more challenging requirements such as guaranteeing logical consistency.

Extensions, as well as strict observation of theoretical concerns, inevitably clash with efficiency concerns, so trade–offs are at hand. In this paper, we outline a few ideas to overcome the efficiency barrier yet offer an increased level of expressivity: in an intuitionistic fuzzy expert system (IFES)[1], intuitionistic fuzzy sets (IFSs) act as *elastic constraints* that allow us to discriminate between the more or less possible values for a variable X, by supplying positive information in favour of the plausible ones (attributing a degree of possibility to the claim that X equals a certain u) and negative information against the implausible ones (ascertaining the degree of certainty, or confidence, that X differs from u). By their distinction between a (weakly connected) membership and a non–membership degree for each element, IFSs are well–positioned to express such bivalent restrictions. In section 2 we compare if–then rule representation by IFSs to other frame-

[1]To avoid confusion, we should stress that the perception of an IFES held in this paper is unrelated to the structure of the same name described by Atanassov in e.g. [1]. We coined the term IFES simply to identify it as a conservative extension of a fuzzy expert system (FES).

works such as certainty factors, interval–valued and second–order fuzzy sets. From the manipulation point of view, inclusion–based approximate reasoning [3, 6], a reasoning methodology based on a measure of fulfilment of the antecedent clause of an if–then rule that allows to alleviate complexity while retaining logical soundness, is described in section 3.

We conclude this section by recalling some preliminary definitions which will be needed throughout the paper. **Intuitionistic fuzzy sets (IFSs)** [1], are a generalization of fuzzy sets. While fuzzy sets give a degree of membership of an element in a given set, IFSs give both a degree of membership and of non–membership. Both of them belong to the interval $[0,1]$, and their sum should not exceed 1. Formally, an IFS A in a universe U was defined as an object of the form $A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\}$, where $\mu_A(u)$ is called the "degree of membership of u in A" and $\nu_A(u)$, the "degree of non-membership of u in A", and where $(\forall u \in U)(\mu_A(u) + \nu_A(u) \leq 1)$. $\pi_A(u) = 1-\mu_A(u)-\nu_A(u)$ is called the "hesitation degree of the element u to A". The class of IFSs in a universe U is denoted $\mathcal{IFS}(U)$. Wang and He in [13], and Deschrijver and Kerre in [8] have shown that IFSs are L–fuzzy sets w.r.t the complete lattice $(L^*, \leq_{L^*})$ defined by $L^* = \{(a_1, a_2) \in [0,1]^2 \mid a_1 + a_2 \leq 1\}$; $(a_1, a_2) \leq_{L^*} (b_1, b_2) \iff a_1 \leq b_1 \wedge a_2 \geq b_2$, and whose units are $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Hence an IFS A in U can also be denoted as an $U \rightarrow L^*$ mapping. In what follows, we adopt the following convention: if $a \in L^*$, then a_1 is the first coordinate and a_2 is the second coordinate of a.

In this context, an **if–then rule** takes the generic form

$$\text{If } X \text{ is } A \text{ then } Y \text{ is } B$$

where X and Y represent an input and an output variable, respectively, and A and B are normalized[2] IFSs in the universe of X and Y. Typically, then, the system is presented with an observation on the input variable of the form "X is A'" with A' not necessarily equalling A. The **Generalized Modus Ponens (GMP)** is an inference rule that allows to combine the rule and the observation on X to yield an observation on Y:

$$\begin{array}{lll} \text{IF} & X \text{ is } A, \text{ THEN} & Y \text{ is } B \\ & X \text{ is } A' & \\ \hline & & Y \text{ is } B' \end{array}$$

The above pattern does not state what the fuzzy restriction B' should be when A, A' and B are given, so an actual implementation of it can take various shapes. In general, however, some elementary conditions (that are in accordance with our view of IFSs as elastic constraints) are imposed on the inference procedure: [3]

$$\begin{array}{ll} \text{A.1} & B \subseteq B' \\ \text{A.2} & A'_1 \subseteq A'_2 \Rightarrow B'_1 \subseteq B'_2 \\ \text{A.3} & A' = A \Rightarrow B' = B \\ \text{A.4} & A' \subseteq A \Rightarrow B' = B \end{array}$$

Subsethood for IFSs is defined, using the ordering $\leq_{L^*}$, by: $A' \subseteq A \iff (\forall u \in U)(A'(u) \leq_{L^*} A(u))$, or alternatively: $A' \subseteq A \iff (\forall u \in U)(\mu_{A'}(u) \leq \mu_A(u) \wedge \nu_{A'}(u) \geq \nu_A(u))$.

One way of obtaining the GMP result is by applying the **Compositional Rule of Inference (CRI)** [6]. Applying this rule for the above FSSs, we obtain:

$$B'(v) = \sup_{u \in U} \mathcal{T}(A'(u), R(u,v)) \tag{1}$$

where sup represents the supremum operation in L^* and $\mathcal{T}$ is an IF (intuitionistic fuzzy) t–norm, i.e. a monotonous, commutative, associative $(L^*)^2 \rightarrow L^*$ mapping satisfying $\mathcal{T}(1^*_L, x) = x, \forall x \in L^*$. This approach is called CRI–GMP. A specific instance of it is (an extension of) Mamdani's approach [11], that defines $\mathcal{T}$ as $\mathcal{T}(x,y) = (\min(x_1, y_1), \max(x_2, y_2))$ and $R(u,v) = (\min(\mu_A(u), \mu_B(v)), \max(\nu_A(u), \nu_B(v)))$. While by this choice a computationally very efficient[3] procedure to obtain B' emerges:

$$\begin{aligned} B'(v) = & \left(\min\left(\sup_{u \in U} \min(\mu'_A(u), \mu_A(u)), \mu_B(v)\right),\right. \\ & \left.\max\left(\inf_{u \in U} \max(\nu'_A(u), \nu_A(u)), \nu_B(v)\right)\right) \end{aligned} \tag{2}$$

[2] An IFS A in U is called normalized if there exists at least one $u \in U$ such that $A(u) = 1_{L^*}$. *Subnormalization*, i.e. the occurence of an IFS that is not normalized, can be seen as a manifestation of (partial) logical conflict.

[3] In general, for finite universes U and V so that $|U| = m$ and $|V| = n$, the CRI–GMP requires $O(mn)$ operations. With Mamdani's choice of parameters, this complexity is reduced to $O(m + n)$ operations.

it can be verified that A.1 and A.4 cannot be maintained. In fact, as Klawonn and Novák remarked in [10], the above calculation rule is not a logical inference, since no logical implication is inside and thus no modus ponens proceeds. They also showed that when applied to a batch of parallel fuzzy rules, Mamdani's method amounts to simple interpolation.

Theoretically, the suitability of a given $(\mathcal{T}, R)$ pair to implement the CRI–GMP can be evaluated with respect to the listed criteria. The following theorem shows that for a given IF t–norm $\mathcal{T}$ satisfying the residuation principle and R defined by $R(u,v) = \mathcal{I}_{\mathcal{T}}(A(u), B(v))$ (where $\mathcal{I}_{\mathcal{T}}$ denotes the residual IF implicator of $\mathcal{T}$, defined by $\mathcal{I}_{\mathcal{T}}(x,y) = \sup\{\gamma \in L^* \mid \mathcal{T}(x,\gamma) \leq_{L^*} y\}$) A.1 through A.4 always hold.

Theorem 1 *Let $\mathcal{T}$ be an IF t–norm satisfying $\mathcal{T}(x,z) \leq_{L^*} y \iff z \leq_{L^*} \mathcal{I}_{\mathcal{T}}(x,y)$ (residuation principle). If B' is defined by $B'(v) = \sup\limits_{u \in U} \mathcal{T}(A'(u), \mathcal{I}_{\mathcal{T}}(A(u), B(v)))$, then A.1–A.4 hold.*

Note that we cannot replace the predicate "satisfying the residuation principle" by "left–continuity" since contrary to fuzzy logic the two notions are not equivalent for IFSs. [7]

2 Vagueness and Uncertainty Modelling

We have mentioned at the beginning that, as information models, fuzzy sets by themselves are too restricted to capture both vagueness and uncertainty in a meaningful way. To amplify that claim, we make the following observation: while $[0,1]$ membership degrees allow us to model positive information flexibly by expressing the extent to which it is *possible* that a variable X assumes a given value u, they take the easy way out when it comes to describing negative information, i.e. to indicate how *certain* it is that X differs from u: indeed, it is customary to assume symmetry between the two notions, in a sense that from knowledge that it is impossible that X equals u, we immediately derive that it is completely certain that X differs from u. Unfortunately, this forces us into a position of unconditional faith in the truthfulness of the information provider, yet, as is well–known, an information source might be discredited for not measuring accurately enough for our purposes, for being defunct or for deliberately lying (if the source is a human), and obviously, we would like this kind of available meta–data to be taken into account and reflected in the result of any inference processes. In [5], we argued that certainty, as opposed to possibility, is a strong and decisive kind of knowledge, and therefore in gathering evidence, it is sensible to be able to first make tentative conjectures without being pinned down on definitive commitments, and later fine–tune observations on the basis of estimated confidence in information sources. In other words, we can have varying degrees of trust in an observer, ranging from unconditional inconfidence to full creditworthiness, and we should be able to model that trust accordingly; which can be done conveniently by letting a certainty degree $\nu(u)$ range between 0 and $1 - \mu(u)$, with $\mu(u)$ interpreted as a possibility degree. This justifies the use of a more general intuitionistic fuzzy, rather than a fuzzy, set as a model of describing observations; in case of full reliability the traditional fuzzy setting, i.e. $\nu(u) = 1 - \mu(u)$, emerges. As fluctuations in certainty may occur in individual facts (assessing the reliability of the observer), as well as in rules (assessing the reliability of the expert who provided the background knowledge of the system to be controlled), the use of IFSs in both premises of the GMP is justified.

It is important to note that IFSs are not the only means of attributing reliability or confidence information to membership degrees. Various frameworks have been conceived to deal with this imperfection, amongst them

- the certainty factor approach: a crisp real–valued number assigned to a fuzzy set expresses the confidence in the implied statement
- interval–valued fuzzy sets (IVFSs): membership degrees are "blurred" by specifying an interval to which the crisp membership degree belongs
- second–order fuzzy sets: an extension of the foregoing approach, where membership degrees themselves become fuzzy sets

We have ranked the alternative approaches according to increasing expressivity: a certainty factor allows only to incorporate general information about the statement as a whole, while the others can differentiate on the level of individual elements of the universe. The second and the third option are treated in detail in a recent book by Mendel [12], and the following observation bears testimony to their close relation to our IFS approach: with every value of $(x_1, x_2) \in L^*$ corresponds a unique interval $[x_1, 1 - x_2]$, and vice versa, and thus IFSs and IVFSs are syntactically equivalent. So rather than setting off on an entirely different course, IFSs can serve to equip the interval–valued approach with a transparent interpretation: the interval's lower limit is identified with the possibility degree $\mu(u)$, the upper limit with the complement to 1 of the necessity degree $\nu(u)$ and the interval length equals the hesitation index $\pi(u)$. We will come back to this equivalence in the next section, where we describe an efficient implementation of the GMP for use in IFESs.

3 Knowledge Manipulation

In section 1, we discussed the CRI–GMP as the principal means to obtain an inference about Y from a single rule linking X and Y, and an observation on X. The aim of this section is twofold: first, by extending the inclusion–based approximate reasoning method from [3], we significantly reduce the complexity of the inference step. Next we generalize the process to a rule base with an arbitrary number of rules. Finally, defuzzification of the result is optional; for a list of appropriate defuzzification techniques (described for IVFSs) we refer to [12].

3.1 Inclusion–Based Reasoning with One Rule

In a nutshell, the rationale behind inclusion–based approximate reasoning is to measure the fulfilment of the antecedent of a rule "X is A" by the observation "X is A'", that is: to check whether A' is a subset of A. For this purpose, rather than sticking to the rigid (crisp) definition of subsethood for IFSs presented in the first section, our attention goes out to $\mathcal{IFS}(U) \times \mathcal{IFS}(U) \to L^*$ mappings Inc, such that the value $Inc(A, B)$ indicates to what extent A is included into B. Our decision to draw inclusion degrees $Inc(A, B)$ from L^*, instead of from the unit interval (it could be argued that the semantics of possibility and confidence attributed to elements of L^* seems at first glance irrelevant to a technical procedure like determining subsethood), rests on the following motivation: since "traditional" subsethood of IFS A into IFS B requires the fulfilment of both $\mu_A \subseteq \mu_B$ and $\nu_B \subseteq \nu_A$[4], two degrees (connected by a weak dependence condition) seem mandatory to reflect the twofold criterion.

At this point, we will not elaborate on the constraints that an IF inclusion indicator should satisfy. As a guideline, the Sinha–Dougherty axioms for fuzzy inclusion indicators and their correction presented in [4], as well as Bustince's investigation into inclusion for IVFSs [2], can be taken.

Now consider we have such a measure (say $Inc(A', A)$): we can use it to transform the consequent IFS B into an appropriate B'. Schematically, this amounts to the following:

$$\begin{array}{llll} \textbf{IF} & X \text{ is } A & \textbf{THEN} & Y \text{ is } B \\ & X \text{ is } A' & & \\ \hline & & & Y \text{ is } f(Inc(A', A), B) \end{array}$$

with f a so–called modification mapping, dependent on the application at hand. Remark how alike this scheme is to similarity–based reasoning, the only difference being that we do not measure the extent to which A and A' are similar: as we argued in [3], symmetry (considered a defining characteristic of a similarity measure) acts counterintuitively, because we compare an observation A' to a reference A and not the other way around; moreover there is no way to satisfy criterium A.4 with the above scheme if symmetry were actually imposed.

Good candidates for the (f, Inc) pair will preferably be such that A.1 through A.4 hold with as little extra conditions added as possible. In addition, we would like to have $Inc(B', B) = Inc(A', A)$, in order that a kind of symmetry between the fulfilment of B' by B and that of A' by A is respected. In [3] and [6] a thorough study

[4]The $\subseteq$ symbol in this formula is meant to represent the inclusion of fuzzy sets

of this problem was conducted for fuzzy rules. It is important that the main result established in those papers continues to hold for the IFS extension, with truly minimal changes:

Theorem 2 *Let $\mathcal{T}$ be an IF t–norm satisfying the residuation principle. If B' represents the result obtained with CRI–GMP based on the $(\mathcal{T}, R)$ pair, with $R(u,v) = \mathcal{I}_{\mathcal{T}}(A(u), B(v))$, i.e. for all $v \in V$, $B'(v) = \sup_{u\in U} \mathcal{T}(A'(u), \mathcal{I}_{\mathcal{T}}(A(u), B(v)))$, and the inclusion measure $Inc_{\mathcal{T}}$ is defined as, for $A', A \in \mathcal{IFS}(U)$, $Inc_{\mathcal{T}}(A', A) = \inf_{u\in U} \mathcal{I}_{\mathcal{T}}(A'(u), A(u))$, then $\mathcal{I}_{\mathcal{T}}(Inc_{\mathcal{T}}(A', A), B(v)) \geq_{L^*} B'(v)$, and $Inc_{\mathcal{T}}(B', B) \geq_{L^*} Inc_{\mathcal{T}}(A', A)$. Additionally, if $(\forall \alpha \in L^*)(\exists v \in V)(B(v) = \alpha)$, then $Inc_{\mathcal{T}}(B', B) = Inc_{\mathcal{T}}(A', A)$.*

This shows that if we put $f = \mathcal{I}_{\mathcal{T}}$, a conclusion entailed by our algorithm is a superset (not necessarily a proper one) of the according CRI–GMP result, which is in accordance with our intuition: when we replace the output of the CRI–GMP by a less specific fuzzy set, the corresponding constraint on the output variable Y will likewise be less strong, since every value of the universe will be assigned at least as high a possibility degree and at least as low a certainty degree by our strategy as by the original CRI–GMP inference mechanism.

It is also worthwhile to relate our approach to one of Bustince's inference procedures for IVFSs [2], paraphrasing the latter to apply to IFSs[5]. It turns out that his method can be identified with an alternative instance of inclusion–based reasoning: after assessing $Inc(A', A)$, the author puts $f(Inc(A', A), B(v)) = (\min(Inc(A', A)_1, \mu_B(v)), \max(Inc(A', A)_2, \nu_B(v)))$; by $Inc(A', A)_1$ and $Inc(A', A)_2$ we denoted the first and second component of $Inc(A', A)$. It is easily verified that with this choice of parameters, A.3 and A.4 hold, but in A.1 and A.2 the inequality signs are reversed, making the system hard to interpret, at least from the point of view of elastic constraints.

We conclude this subsection with an observation about complexity. Since the inclusion–based approach bypasses the expensive supremum operation in formula (1), its complexity for finite universes equals that of Mamdami's approach.

[5]By the equivalence of both structures, this is a legitimate conversion.

3.2 Inclusion–Based Reasoning with Parallel Rules

We now consider the case of n parallel rules:

IF X is A_1 **THEN** Y is B_1
IF X is A_2 **THEN** Y is B_2
...
IF X is A_n **THEN** Y is B_n

where we assume that for $i \neq j$, $A_i \not\subseteq A_j$. In order to perform inference on the basis of a block of fuzzy rules, we need a mechanism to somehow execute the rules in parallel. To safeguard its logical soundness, we impose two criteria on this aggregation procedure: by *coherency* is implied that for every collection of parallel fuzzy rules, when the observation exactly matches the antecedent of one of the rules, the inference outcome equals the consequent of this rule. By *consistency* we mean that a system may only return normalized IFSs. In [6], a detailed study of aggregation operators and the fulfilment of coherency and consistency was conducted, to the conclusion that it proves very hard to join these two characteristics into a single system. The main reasons for this deficiency are the following:

- Since different rules can have varying degrees of influence on the final outcome, even in the case of a perfect match between the observation and one of the rule antecedents, the effect of the other rules can shift the inference result away from the consequent of that rule (thus violating coherency).

- Typically, rules, to a certain extent, conflict one another. When merging their individual outcomes into the final result by means of an aggregation operator, this partial conflict is reflected by subnormalization (thus violating consistency).

In the light of these observations, Dvořák's rule preselection scheme [9] may be considered an acceptable compromise:

1. For $i = 1, ..., n$, calculate $\alpha_i = Inc(A', A_i)$
2. Choose rule l with the highest value of α_l
3. Compute the conclusion B' by means of, for $v \in V$: $B'(v) = f(\alpha_l, B_l(v))$

The basic idea behind rule preselection is to extract from the knowledge base only that rule which is best in accordance with the observation. It is assumed that this rule provides us with sufficient information about what the conclusion should look like, even if the balancing behaviour of a "genuine" aggregation strategy (i.e. taking into account all partial results) is disposed of in this way. Dvořák's algorithm has the advantage of always being coherent (due to the restrictions we imposed on the rule base) and consistent *provided no two rules yield the same highest inclusion degree*: if the latter occurs, the algorithm does not make it clear which rule to choose. Dvořák proposes an auxiliary criterion (e.g. the Hausdorff distance between the kernels of A' and the involved rule antecedents) to cut the knot. An additional difficulty presents itself when we want to apply Dvořák's method to parallel rules in an IFES: since the degrees α_i are elements of L^*, which is only partially ordered, the maximum calculation in step 2 can not always be performed. How to rank the inclusion degrees depends strongly on the indicator used, and will be a subject of future study.

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References

[1] K. T. Atanassov (1999). Intuitionistic Fuzzy Sets, Physica-Verlag, Heidelberg, New York.

[2] H. Bustince (2000). Indicator of inclusion grade for interval–valued fuzzy sets. Application to approximate reasoning based on interval–valued fuzzy sets. In *International Journal of Approximate Reasoning* **23**, pages 137–209.

[3] C. Cornelis, E.E. Kerre (2001). Inclusion-based approximate reasoning. In *Lecture Notes in Computer Science 2074*, (V. Alexandrov, J. Dongarra, B. Juliano, R. Renner, C. Tan, eds.), Springer-Verlag, pages 200-210.

[4] C. Cornelis, C. Van Der Donck, E.E. Kerre (2002). Sinha–Dougherty approach to the fuzzification of set inclusion revisited. Accepted to Fuzzy Sets and Systems.

[5] C. Cornelis, E.E. Kerre (2002). Reliability of Information: Motivation for an Intuitionistic Theory of Possibility. Submitted to Fuzzy Sets and Systems.

[6] C. Cornelis, E.E. Kerre (2002). A fuzzy inference methodology based on the fuzzification of set inclusion. In *Innovations in Intelligent Systems* (A. Abraham, B.Nath, eds.), Studies in Fuzziness and Soft Computing Series, Springer-Verlag, in press.

[7] G. Deschrijver, C. Cornelis, E. E. Kerre (2002). On the representation of intuitionistic fuzzy t-norms, in preparation.

[8] G. Deschrijver, E. E. Kerre (2002). On the relationship between some extensions of fuzzy set theory. Accepted to Fuzzy Sets and Systems.

[9] A. Dvořák (2000). On preselection of rules in fuzzy logic deduction. In *International Journal of Uncertainty, Fuzziness and Knowledge–Based Systems*, **8(5)**, pages 563-573.

[10] F. Klawonn, V. Novak (1996). The relation between inference and interpolation in the framework of fuzzy systems. In *Fuzzy Sets and Systems*, **81**, pages 331-354.

[11] E.H. Mamdani, S. Assilian (1975). An experiment in linguistic synthesis with a fuzzy logic controller. In *International Journal of Man-Machine Studies*, **7**, pages 1-13.

[12] J.M. Mendel (2001). Uncertain rule–based fuzzy logic systems, Prentice Hall PTR, Upper Saddle River, New Jersey.

[13] G. Wang, Y. He (2000). Intuitionistic fuzzy sets and L-fuzzy sets. In *Fuzzy Sets and Systems*, **110(2)**, pages 271-274.