A Fuzzy Formal Logic for Interval-valued Residuated Lattices

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Abstract

Fuzzy formal logics were introduced in order to handle graded truth values instead of only 'true' and 'false'. A wide range of such logics were introduced successfully, like Monoidal T-norm based Logic, Basic Logic, Gödel Logic, Łukasiewicz Logic etc. However, in general, fuzzy set theory is not only concerned with vagueness, but also with uncertainty. A possible solution is to use intervals instead of real numbers as membership values. In this paper, we present an approach with triangle algebras, which are algebraic characterizations of interval-valued residuated lattices. The variety of these structures corresponds in a sound and complete way to a logic that we introduce, called Triangle Logic (in the same way as, e.g., BL-algebras and Basic Logic). We will show that this truth-functional approach, along with the residuation principle, has some consequences that seem to obstruct an easy and proper interpretation for the semantics of Triangle Logic.

Keywords: Fuzzy logic, Interval-valued fuzzy set theory, Residuated lattices.

1 Introduction and Preliminaries

1.1 Fuzzy formal logics

In classical logic[1], a formula is provable if, and only if, its value (under every evaluation) is 1 in every Boolean algebra (this is called soundness and completeness). Moreover, it suffices to consider the only linear Boolean algebra, which has two elements (0 and 1). We can therefore say that classical logic is a two-valued logic. The value 0 stands for 'false', 1 for 'true'. However, in many cases it may be preferable to work with graded truth values, e.g. for propositions like 'a cat is a small animal', 'Peter has dark hair' or 'the weather is bad today'. This is the reason why fuzzy (formal) logics, which are generalizations of classical logic, were introduced. One of the most general examples is Höhle's Monoidal Logic (ML) [9]. In this logic, a formula is provable iff its value is 1 in every residuated lattice. Recall that a residuated lattice is a structure $\mathcal{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ in which $\sqcap, \sqcup, *$ and $\Rightarrow$ are binary operators on L and

- $(L, \sqcap, \sqcup)$ is a bounded lattice with 0 as smallest and 1 as greatest element,
- $*$ is commutative and associative, with 1 as neutral element, and
- $x * y \leq z$ iff $x \leq y \Rightarrow z$ for all x, y and z in L (residuation principle).

This allows one to assign graded truth values to formulas. Of course, not every formula that is provable in classical logic, is provable in ML too. However, we can add axioms to obtain intermediary logics. An interesting example is Esteva and Godo's Monoidal T-norm based Logic (MTL) [7]. For this fuzzy logic, the so-called prelinearity axiom $((\phi \rightarrow \psi) \vee (\psi \rightarrow \phi))$ is added to the axioms of ML. As for all axiomatic extensions of ML, MTL is sound and complete w.r.t. the associated variety of algebraic structures (in this case: MTL-algebras, which are residuated lattices in which $(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$ holds for all x and y). Moreover, because of the prelinearity MTL is also sound and complete w.r.t. linear MTL-algebras[2][7]. The latter property is preserved for axiomatic extensions of MTL. For example, Basic Logic (BL) [8] is sound and complete w.r.t. linear BL-algebras, Łukasiewicz logic [11] is sound and complete w.r.t. Wajsberg algebras,

[1] In this paper we only deal with propositional logics. Therefore we will not write this explicitly.

[2] Recall that linear MTL-algebra is the same as linear residuated lattices, because linearity is a stronger property than prelinearity in residuated lattices.

and classical logic is sound and complete w.r.t. the linear Boole algebra. In some of these cases, it is even possible to further improve this result in the sense that one only has to take into account the algebraic structures on the unit interval. Two examples are MTL and BL, which are sound and complete w.r.t. MTL-algebras on $[0, 1]$ and BL-algebras on $[0, 1]$, respectively [1, 10]. These are exactly the structures induced by left-continuous t-norms and continuous t-norms, respectively. So we can speak of BL as the logic of continuous t-norms on $[0, 1]$.

1.2 Intervals as Truth Values

In order to handle also uncertainty together with vagueness, it seems a good idea to use closed subintervals of $[0, 1]$ instead of real numbers in $[0, 1]$. Interval-valued truth degrees have been widely adopted in knowledge-based systems [6]. This is due to the relative efficiency of operations defined on them, as well as to the fact that they carry an attractive and straightforward semantical interpretation as partial, or incomplete, truth values, i.e., they exhibit a lack of knowledge about a formula's exact truth value; the wider the interval, the greater the uncertainty.

The lattice $\mathcal{L}^I = (L^I, \sqcap, \sqcup)$ that contains these closed subintervals is shown graphically in Figure 1 and defined by

$$L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 \leq x_2\},$$
$$[x_1, x_2] \sqcap [y_1, y_2] = [\min(x_1, y_1), \min(x_2, y_2)],$$
$$[x_1, x_2] \sqcup [y_1, y_2] = [\max(x_1, y_1), \max(x_2, y_2)].$$

Its partial ordering $\leq_{L^I}$ is given by componentwise extension of $\leq$,

$$[x_1, x_2] \leq_{L^I} [y_1, y_2] \iff x_1 \leq y_1 \text{ and } x_2 \leq y_2.$$

We use the 'componentwise' ordering, which is in our opinion the most natural one. Indeed, it satisfies $x \leq y$ iff $x \sqcap y = x$ iff $x \sqcup y = y$, and $x \sqcap y = \{\min(x_0, y_0) \mid (x_0, y_0) \in x \times y\}$ and $x \sqcup y = \{\max(x_0, y_0) \mid (x_0, y_0) \in x \times y\}$.

One way to interpret these intervals is to see them as a kind of confidence intervals. For example, a truth value of $[a, b]$ might mean that the exact, but unknown, truth value is definitely greater than or equal to a and smaller than or equal to b (100%-confidence interval).

Research on fuzzy formal logics has centered on prelinear residuated structures. However, while prelinearity holds in every residuated lattice $([0, 1], \min, \max, *, \Rightarrow, 0, 1)$, it is not necessarily preserved for closed intervals of a bounded lattice $\mathcal{L}$; for example, it was shown in [2] that no MTL-algebra exists on the lattice $\mathcal{L}^I$.

The goal of this paper is to develop a logic that formally characterizes tautologies (true formulas) in

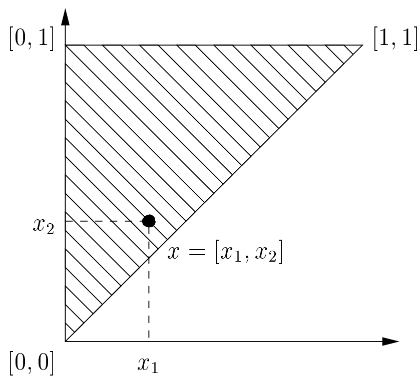

Figure 1: The lattice $\mathcal{L}^I$

interval-valued residuated lattices (IVRLs). Monoidal Logic, which corresponds to the complete class of residuated lattices, is too general for our purposes, and we need to extend it with suitable axioms to replace prelinearity. To achieve this, we propose the use of modal-like operators.

In Section 2 we first recall triangle algebras and their relationship with residuated lattices on interval-valued algebraic structures [12]. In Section 3 we will introduce Triangle Logic and show soundness and completeness w.r.t. triangle algebras. We round up the paper with some remarks about the suitability of TL for modelling reasoning with uncertain propositions, and a conclusion.

2 Triangle algebras

In view of the considerations in Section 1, we defined [12] algebraic structures that should satisfy as many properties as possible of residuated lattices on L^I, like MTL-algebras are the algebraic structures that satisfy as many properties as possible of residuated lattices on $[0, 1]$. These structures are called triangle algebras. Before we give their definition and discuss triangle algebras, we give some additional definitions:

Definition 1 Given a lattice $\mathcal{L} = (L, \sqcap, \sqcup)$, its triangularization $\mathbb{T}(\mathcal{L})$ is the structure $\mathbb{T}(\mathcal{L}) = (Int(\mathcal{L}), \bigsqcap, \bigsqcup)$ defined by

- $Int(\mathcal{L}) = \{[x_1, x_2] \mid (x_1, x_2) \in L^2 \text{ and } x_1 \leq x_2\}$
- $[x_1, x_2] \bigsqcap [y_1, y_2] = [x_1 \sqcap y_1, x_2 \sqcap y_2]$
- $[x_1, x_2] \bigsqcup [y_1, y_2] = [x_1 \sqcup y_1, x_2 \sqcup y_2]$

The set $D_{\mathcal{L}} = \{[x, x] \mid x \in L\}$ is called the diagonal of $\mathbb{T}(\mathcal{L})$.

The first and the second projection pr_1 and pr_2 are the mappings from $Int(\mathcal{L})$ to L, defined by $pr_1([x_1, x_2]) =$

x_1 and $pr_2([x_1, x_2]) = x_2$, for all $[x_1, x_2]$ in $Int(\mathcal{L})$. We also define the mappings p_v and p_h from $Int(\mathcal{L})$ to $D_{\mathcal{L}}$ as $p_v([x_1, x_2]) = [x_1, x_1]$ and $p_h([x_1, x_2]) = [x_2, x_2]$, for all $[x_1, x_2]$ in $Int(\mathcal{L})$.
A t-norm $\mathcal{T}$ on a triangularization of a bounded lattice is called t-representable if $pr_1(\mathcal{T}([x_1, x_2], [y_1, y_2]))$ is independent of x_2 and y_2, and $pr_2(\mathcal{T}([x_1, x_2], [y_1, y_2]))$ is independent of x_1 and y_1.
An interval-valued residuated lattice (IVRL) is a residuated lattice $(Int(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_{\odot}, [0,0], [1,1])$ on the triangularization $\mathbb{T}(\mathcal{L})$ of a bounded lattice $\mathcal{L}$, in which the diagonal $D_{\mathcal{L}}$ is closed under $\odot$ and $\Rightarrow_{\odot}$, i.e., $[x_1, x_1] \odot [y_1, y_1] \in D_{\mathcal{L}}$ and $[x_1, x_1] \Rightarrow_{\odot} [y_1, y_1] \in D_{\mathcal{L}}$ for x_1, y_1 in L.

Note for example that $\mathcal{L}^I$ is the triangularization of $([0,1], \min, \max)$.

Example 1 *It was shown in [3] that if T is a left-continuous t-norm on $[0,1]$, then for each α in $[0,1]$, the mapping $\mathcal{T}_{T,\alpha}$ defined by, for $x = [x_1, x_2]$ and $y = [y_1, y_2]$ in L^I, $\mathcal{T}_{T,\alpha}(x, y) =$*

$$[T(x_1, y_1), \max(T(\alpha, T(x_2, y_2)), T(x_1, y_2), T(x_2, y_1))],$$

is a t-norm on L^I, with residual implicator[3] $\mathcal{I}_{\mathcal{T}_{T,\alpha}}(x, y)$

$$= [\min(I_T(x_1, y_1), I_T(x_2, y_2)), \min(I_T(T(x_2, \alpha), y_2), I_T(x_1, y_2))],$$

that induces a residuated lattice on $\mathcal{L}^I$. Because also $\mathcal{T}_{T,\alpha}([x,x],[y,y]) = [T(x,y), T(x,y)]$ and $\mathcal{I}_{\mathcal{T}_{T,\alpha}}([x,x],[y,y]) = [I_T(x,y)), I_T(x,y)]$ for all x and y in $[0,1]$, $(L^I, \sqcap, \sqcup, \mathcal{T}_{T,\alpha}, \mathcal{I}_{\mathcal{T}_{T,\alpha}}, [0,0], [1,1])$ is an IVRL.

Two important values of α can be distinguished:

- *If $\alpha = 1$, we obtain t-representable t-norms on $\mathcal{L}^I$: $\mathcal{T}_{T,1}(x, y) = [T(x_1, y_1), T(x_2, y_2)]$, which can be seen as the straightforward (and most commonly used) extension of T to $\mathcal{L}^I$. These t-norms on $\mathcal{L}^I$ are characterized by the property $p_h(\mathcal{T}(x, y)) = \mathcal{T}(p_h(x), p_h(y))$.*
- *If $\alpha = 0$, we obtain pseudo t-representable t-norms on $\mathcal{L}^I$: $\mathcal{T}_{T,0}(x, y) = [T(x_1, y_1), \max(T(x_1, y_2), T(x_2, y_1))]$. These t-norms are inherently more complex than their t-representable counterparts, but, as we shall see at the end of this section, satisfy more relevant properties.*

Definition 2 *[12]* A triangle algebra is a structure $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, in which $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice, in which ν and μ are binary operators and u a constant, and in which the following conditions hold:

$T.1$	$\nu x \leq x$,	$T.1'$	$x \leq \mu x$,
$T.2$	$\nu x \leq \nu\nu x$,	$T.2'$	$\mu\mu x \leq \mu x$,
$T.3$	$\nu(x \sqcap y) = \nu x \sqcap \nu y$,	$T.3'$	$\mu(x \sqcap y) = \mu x \sqcap \mu y$,
$T.4$	$\nu(x \sqcup y) = \nu x \sqcup \nu y$,	$T.4'$	$\mu(x \sqcup y) = \mu x \sqcup \mu y$,
$T.5$	$\nu 1 = 1$,	$T.5'$	$\mu 0 = 0$,
$T.6$	$\nu u = 0$,	$T.6'$	$\mu u = 1$,
$T.7$	$\nu\mu x = \mu x$,	$T.7'$	$\mu\nu x = \nu x$,

$T.8$	$\nu(x \Rightarrow y) \leq \nu x \Rightarrow \nu y$,
$T.9$	$(\nu x \Leftrightarrow \nu y) * (\mu x \Leftrightarrow \mu y) \leq (x \Leftrightarrow y)$,
$T.10$	$\nu x \Rightarrow \nu y \leq \nu(\nu x \Rightarrow \nu y)$.

The set $\{x \in A \mid \nu x = x\}$ is called the set of exact elements $E(\mathcal{A})$ of the triangle algebra $\mathcal{A}$.

We use the modal-like operators ν and μ in order to represent the lower and upper bound of an interval. The set of exact elements corresponds to the diagonal of an IVRL.

Proposition 1 *[12] Let $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra. Then $E(\mathcal{A})$ is the direct image of A under ν, as well as under μ. Moreover, this set is invariant under ν and μ, and contains 0 and 1, but not u (unless in the trivial case when $|A| = 1$). It is closed under $\sqcap$, $\sqcup$, $*$ and $\Rightarrow$.*

The next theorem [12] establishes the equivalence between IVRLs and triangle algebras:

Theorem 1 *Every triangle algebra $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is isomorphic to a triangle algebra $(Int(\mathcal{E}(\mathcal{A})), \bigsqcap, \bigsqcup, \odot, \Rightarrow_{\odot}, p_v, p_h, [0,0], [0,1], [1,1])$ where $(Int(\mathcal{E}(\mathcal{A})), \bigsqcap, \bigsqcup, \odot, \Rightarrow_{\odot}, [0,0], [1,1])$ is an IVRL.*
Conversely, if $(A, \sqcap, \sqcup, \odot, \Rightarrow_{\odot}, [0,0], [1,1])$ is an IVRL and ν and μ are defined by $\nu[x_1, x_2] = [x_1, x_1]$ and $\mu[x_1, x_2] = [x_2, x_2]$, then $(A, \sqcap, \sqcup, \odot, \Rightarrow_{\odot}, \nu, \mu, [0,0], [0,1], [1,1])$ is a triangle algebra.

This theorem shows that triangle algebras are a good characterization for IVRLs. However, IVRLs are still more general than residuated lattices on L^I. For example, the diagonal of an IVRL needs not be linear. We can impose the property that the diagonal should be prelinear, but it is currently unknown if this is enough to guarantee that all identities that are true in IVRLs with linear diagonal are also true in every IVRL with prelinear diagonal. This property would be comparable to the fact that all the identities that are true in linear residuated lattices are also true in prelinear residuated lattices [7].

[3] If T is a t-norm on a bounded lattice $(L, \sqcap, \sqcup)$, its residual implicator I_T is defined as $I_T(x, y) = \sup\{z \in L \mid T(x, z) \leq y\}$ for all x and y in L, if the supremum exists. It always exists in complete lattices and residuated lattices (provided that $*$ is the t-norm).

3 Triangle Logic

In this section we translate the defining properties of triangle algebras into logical axioms, and show that the resulting logic TL is sound and complete w.r.t. the variety of triangle algebras.

The language of TL consists of countably many proposition variables (p_1, p_2,...), the constants $\overline{0}$ ('falsity') and $\perp$ ('uncertainty'), the unary operators $\Box$ ('necessity'), $\Diamond$ ('possibility'), the binary operators $\wedge$ ('weak conjunction'), $\vee$ ('disjunction'), & ('strong conjunction'), $\rightarrow$ ('implication'), and finally the auxiliary symbols '(' and ')'. Formulas are defined inductively: proposition variables, $\overline{0}$ and $\perp$ are formulas; if ϕ and ψ are formulas, then so are $(\phi\wedge\psi)$, $(\phi\vee\psi)$, $(\phi\&\psi)$, $(\phi \rightarrow \psi)$, $(\Box\psi)$ and $(\Diamond\psi)$.

In order to avoid unnecessary brackets, we agree on the following priority rules:

- unary operators always take precedence over binary ones,
- among the binary operators, & has the highest priority; furthermore $\wedge$ and $\vee$ take precedence over $\rightarrow$,
- the outermost brackets are not written.

We also introduce some useful shorthand notations: $\overline{1}$ for $\overline{0} \rightarrow \overline{0}$, $\neg\phi$ for $\phi \rightarrow \overline{0}$ and $\phi \leftrightarrow \psi$ for $(\phi \rightarrow \psi)\wedge(\psi \rightarrow \phi)$ for formulas ϕ and ψ.

The axioms of TL are those of ML (Monoidal Logic)

$$
\begin{array}{ll}
ML.1 & (\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi)),\\
ML.2 & \phi \rightarrow (\phi\vee\psi),\\
ML.3 & \psi \rightarrow (\phi\vee\psi),\\
ML.4 & (\phi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\phi\vee\psi) \rightarrow \chi)),\\
ML.5 & (\phi\wedge\psi) \rightarrow \phi,\\
ML.6 & (\phi\wedge\psi) \rightarrow \psi,\\
ML.7 & (\phi\&\psi) \rightarrow \phi,\\
ML.8 & (\phi\&\psi) \rightarrow (\psi\&\phi),\\
ML.9 & (\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow \chi) \rightarrow (\phi \rightarrow (\psi\wedge\chi))),\\
ML.10 & (\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi\&\psi) \rightarrow \chi),\\
ML.11 & ((\phi\&\psi) \rightarrow \chi) \rightarrow (\phi \rightarrow (\psi \rightarrow \chi)),\\
ML.12 & \overline{0} \rightarrow \phi,
\end{array}
$$

complemented with axioms corresponding to T.1–T.10 and T.1'–T.7':

$$
\begin{array}{ll}
TL.1 & \Box\phi \rightarrow \phi,\\
TL.1' & \phi \rightarrow \Diamond\phi,\\
TL.2 & \Box\phi \rightarrow \Box\Box\phi,\\
TL.2' & \Diamond\Diamond\phi \rightarrow \Diamond\phi,\\
TL.3 & (\Box\phi\wedge\Box\psi) \rightarrow \Box(\phi\wedge\psi),\\
TL.3' & (\Diamond\phi\wedge\Diamond\psi) \rightarrow \Diamond(\phi\wedge\psi),\\
TL.4 & \Box(\phi\vee\psi) \rightarrow (\Box\phi\vee\Box\psi),\\
TL.4' & \Diamond(\phi\vee\psi) \rightarrow (\Diamond\phi\vee\Diamond\psi),\\
TL.5 & \Box\overline{1},\\
TL.5' & \neg\Diamond\overline{0},\\
TL.6 & \neg\Box\perp,\\
TL.6' & \Diamond\perp,\\
TL.7 & \Diamond\phi \rightarrow \Box\Diamond\phi,\\
TL.7' & \Diamond\Box\phi \rightarrow \Box\phi,\\
TL.8 & \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi),\\
TL.9 & (\Box\phi \leftrightarrow \Box\psi)\&(\Diamond\phi \leftrightarrow \Diamond\psi) \rightarrow (\phi \leftrightarrow \psi),\\
TL.10 & (\Box x \rightarrow \Box y) \rightarrow \Box(\Box x \rightarrow \Box y).
\end{array}
$$

The deduction rules are modus ponens (MP, from ϕ and $\phi \rightarrow \psi$ infer ψ), generalization (G, from ϕ infer $\Box\phi$) and monotonicity of $\Diamond$ (M$\Diamond$, from $\phi \rightarrow \psi$ infer $\Diamond\phi \rightarrow \Diamond\psi$).

The consequence relation $\vdash$ is defined as follows, in the usual way. Let V be a theory, i.e., a set of formulas in TL. A (formal) proof of a formula ϕ in V is a finite sequence of formulas with ϕ at its end, such that every formula in the sequence is either an axiom of TL, a formula of V, or the result of an application of an inference rule to previous formulas in the sequence. If a proof for ϕ exists in V, we say that ϕ can be deduced from V and we denote this by $V \vdash \phi$.

Note that TL.5 is in fact superfluous, as it immediately follows from $\emptyset \vdash \overline{1}$ and generalization; we include it here to obtain full correspondence with Definition 2.

To show that this logic is sound and complete w.r.t. the variety of triangle algebras, we use the same approach as for, e.g. ML [9], MTL [7] and BL [8]. We first establish some intermediate results. For a theory V, and formulas ϕ and ψ in TL, denote $\phi \sim_V \psi$ iff $V \vdash \phi \rightarrow \psi$ and $V \vdash \psi \rightarrow \phi$ (this is also equivalent with $V \vdash \phi \leftrightarrow \psi$). Note that $\sim_V$ is an equivalence relation on the set of formulas. Moreover, it is a congruence w.r.t. $\Box$, $\Diamond$, $\wedge$, $\vee$, & and $\rightarrow$; this means that the results of the application of these connectives are equivalent whenever the arguments are equivalent.As a consequence, we can meaningfully consider the structure $(A_V, \wedge_V, \vee_V, \&_V, \rightarrow_V, \Box_V, \Diamond_V, [\overline{0}]_V, [\perp]_V, [\overline{1}]_V)$, in which

- A_V is the set of equivalence classes of $\sim_V$, i.e.

$A/_{\sim_V}$,

- $\wedge_V$ is the binary operation on A_V that maps $([\phi]_V, [\psi]_V)$ to $[\phi\wedge\psi]_V$,
- $\vee_V$ is the binary operation on A_V that maps $([\phi]_V, [\psi]_V)$ to $[\phi\vee\psi]_V$,
- $\&_V$ is the binary operation on A_V that maps $([\phi]_V, [\psi]_V)$ to $[\phi\&\psi]_V$,
- $\rightarrow_V$ is the binary operation on A_V that maps $([\phi]_V, [\psi]_V)$ to $[\phi \rightarrow \psi]_V$,
- $\Box_V$ is the unary operation on A_V that maps $[\phi]_V$ to $[\Box\phi]_V$,
- $\Diamond_V$ is the unary operation on A_V that maps $[\phi]_V$ to $[\Diamond\phi]_V$,
- $[\overline{0}]_V$, $[\bot]_V$, $[\overline{1}]_V$ are the elements of A_V that contain $\overline{0}$, $\bot$ and $\overline{1}$ resp.

Proposition 2 The structure $(A_V, \wedge_V, \vee_V, \&_V, \rightarrow_V, \Box_V, \Diamond_V, [\overline{0}]_V, [\bot]_V, [\overline{1}]_V)$ is a triangle algebra.

If $V = \emptyset$, this structure is called the Lindenbaum-algebra of TL.

Definition 3 *Let $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra and V a theory. An $\mathcal{A}$-evaluation is a mapping e from the set of formulas of TL to A that satisfies, for each two formulas ϕ and ψ:*

- $e(\phi \wedge \psi) = e(\phi) \sqcap e(\psi)$,
- $e(\phi \vee \psi) = e(\phi) \sqcup e(\psi)$,
- $e(\phi\&\psi) = e(\phi) * e(\psi)$,
- $e(\phi \rightarrow \psi) = e(\phi) \Rightarrow e(\psi)$,
- $e(\Box\phi) = \nu e(\phi)$,
- $e(\Diamond\phi) = \mu e(\phi)$,
- $e(\overline{0}) = 0$ *and*
- $e(\bot) = u$.

If an $\mathcal{A}$-evaluation e satisfies $e(\chi) = 1$ for every χ in V, it is called an $\mathcal{A}$-model for V.

Theorem 2 (**Soundness and completeness of TL**) A formula ϕ can be deduced from a theory V in TL iff for every triangle algebra $\mathcal{A}$ and for every $\mathcal{A}$-model e of V, $e(\phi) = 1$.

Theorem 2 implies similar results for more specific logics.

- For example, if we add $\mu(x * y) = \mu x * \mu y$ to the conditions of a triangle algebra and $\Diamond\phi\&\Diamond\psi \rightarrow \Diamond(\phi\&\psi)$ to the axioms of TL, then we can obtain a valid theorem by replacing 'triangle algebra' and 'Triangle Logic' in the formulation of Theorem 2 by the new algebra and logic. This property implies that (in terms of IVRL) the second component of $[x_1, x_2] * [y_1, y_2]$ is independent of x_1 and y_1. This means that we can use this property to characterize IVRLs with t-representable t-norms by triangle algebras satisfying $\mu(x * y) = \mu x * \mu y$.
- Another interesting example is $x = \neg\neg x$, connected to the axiom $\neg\neg\phi \rightarrow \phi$. The only involutive t-norms of the form $\mathcal{T}_{T,\alpha}$ are the pseudo t-representable ones [3]. More generally, in an involutive triangle algebra, it can be verified that $u * u = 0$.
- As a final example, we can add $(\nu x \Rightarrow \nu y) \sqcup (\nu y \Rightarrow \nu x) = 1$ to the conditions of a triangle algebra (remark that this property is always satisfied for triangle algebras on $\mathcal{L}^I$, because its diagonal is linearly ordered). If also we add $(\Box\phi \rightarrow \Box\psi)\vee(\Box\psi \rightarrow \Box\phi)$ to the axioms of TL, then again we obtain a valid theorem by replacing 'triangle algebra' and 'Triangle Logic' in Theorem 2 by the new algebra and logic. In this case $(E(\mathcal{A}), \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is an MTL-algebra (prelinear residuated lattice). This means that it is a subalgebra of the direct product of a system of linearly ordered residuated lattices [9]. Using this property, a stronger form of completeness, called chain completeness, can be proven for MTL: a formula ϕ can be deduced from a theory V in MTL iff for every linearly ordered MTL-algebra $\mathcal{A}$ and for every $\mathcal{A}$-model e of V, $e(\phi) = 1$. Similar results hold for subvarieties of the variety of MTL-algebras and their corresponding logics (e.g. BL and Ł). We would like to find analogous theorems for triangle algebras (and subvarieties) too, but at this moment it is still an open question if every triangle algebra satisfying $(\nu x \Rightarrow \nu y) \sqcup (\nu y \Rightarrow \nu x) = 1$ is a subalgebra of the direct product of a system of triangle algebras with linearly ordered diagonal.

Note 1 Triangle Logic is a truth-functional logic: the truth degree of a compound proposition is determined by the truth degree of its parts. This causes some counterintuitive results, if we want to interpret the element $[0, 1]$ of an IVRL as uncertainty. For example: suppose we don't know anything of the truth value of propositions p and q, i.e., $v(p) = v(q) = [0, 1]$. Then yet the implication $p \rightarrow q$ is definitely valid: $v(p \rightarrow q) = v(p) \Rightarrow v(q) = [1, 1]$. However, if

$\neg[0,1] = [0,1]$ [4] (which is intuitively preferable, since the negation of an uncertain proposition is still uncertain), then we can take $q = \neg p$, and obtain that $p \to \neg p$ is true. Or, equivalently (using the residuation principle), that $p\&p$ is false. This does not seem intuitive, as one would rather expect $p\&p$ to be uncertain if p is uncertain.

Another consequence of $[0,1] \Rightarrow [0,1] = [1,1]$ is that it is impossible to interpret the intervals as a set in which the 'real' (unknown) truth value is contained, and $X \Rightarrow Y$ as the smallest closed interval containing every $x \Rightarrow y$, with x in X and y in Y (as in [6]). Indeed: $1 \in [0,1]$ and $0 \in [0,1]$, but $1 \Rightarrow 0 = 0 \notin [1,1]$.

On the other hand, for t-norms it is possible that $X * Y$ is the smallest closed interval containing every $x * y$, with x in X and y in Y, but only if they are t-representable (described by the axiom $\mu(x * y) = \mu x * \mu y$). However, in this case $\neg[0,1] = [0,0]$, which does not seem intuitive ('the negation of an uncertain proposition is absolutely false').

These considerations seem to suggest that Triangle Logic is not suitable to reason with uncertainty. This does not mean that intervals are not a good way for representing degrees of imprecise knowledge, only that they are not suitable as truth values in a truth functional logical calculus when we interpret them as expressing imprecision. It might even be impossible to model uncertainty as a truth value in any truth-functional logic. This question is discussed in [4, 5]. However, nothing prevents the intervals in Triangle Logic from having more adequate interpretations.

4 Conclusion

In this paper we explained why we wanted to construct a (family of) fuzzy formal logic(s) with intervals as truth values. We showed how triangle algebras can be used to achieve this goal, as they characterize interval-valued residuated lattices. We introduced Triangle Logic and proved soundness and completeness w.r.t. triangle algebras, and thus w.r.t. interval-valued residuated lattices. Finally, we made some observations which seem to limit the usefulness of Triangle Logic, and indeed of any truth-functional logic, to model reasoning about uncertain propositions.

Acknowledgement

Bart Van Gasse and Chris Cornelis would like to thank the Research Foundation–Flanders for funding their research.

[4] This is for example the case if $\neg$ is involutive.

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