Application of Extended Modal Operators as a Tool for Constructing Inclusion Indicators over Intuitionistic Fuzzy Sets

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Abstract

In this paper, we revisit the notion of inclusion for intuitionistic fuzzy sets (IFSs). Applying the extended modal logic operators D_α *over IFSs, we construct a new class of two–valued inclusion indicators; we compare it to the original proposal from [1]; and finally we exploit it to define an intuitively meaningful notion of graded inclusion indicators over IFSs, parallelling and clarifying earlier work on (direct) generalizations of fuzzy inclusion measures.*

1 Introduction

Intuitionistic fuzzy set (IFS) theory, first described in [1], basically enriches Zadeh's fuzzy set theory [17] with a notion of indeterminacy. While in the latter, membership degrees, identifying the degree to which an object satisfies a given property are taken to be exact, in the former extra information in the guise of a non–membership degree is permitted to address a commonplace feature of uncertainty. Imagine, for instance, a voting procedure in which delegates have to express their feelings w.r.t. a number of proposals. It is obvious that while one can be in favour or in disfavour of a proposal to a certain extent, one can also abstain from the vote; an attitude inspired by, e.g., a lack of background or interest, or simply because no obvious arguments for or against the cause at stake have been raised.

In brief, IFS theory does not insist that membership and non–membership to a set be strictly complementary. In an IFS A defined in a universe X, alongside a *membership degree* $\mu_A(x)$ of x to A, we also distinguish a *non–membership degree* $\nu_A(x)$, such that $\mu_A(x) + \nu_A(x) \leq 1$. Note that a fuzzy set in X is then just an IFS for which $\mu_A(x) + \nu_A(x) = 1$ holds for every x. The degree $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ quantifies the *degree of indeterminacy* associated with element x and predicate A.

To make IFS theory operational, it must be furnished with *faithful* and *adequate* extensions of corresponding fuzzy set operations: that is, when applied to fuzzy sets these extensions must yield the same result as before, yet they should also take into account the particular challenges raised by IFS theory. This process is illustrated nicely for instance by the various approaches to the definition and classification of basic set–theoretical operations such as complement, intersection and union of IFSs: "standard" operations were introduced in [1] right at the outset of the theory, but later on various alternatives and modifications (see e.g. [3, 8, 10]) refining and enriching the original proposal have sprung up.

In [1], a notion of subsethood of IFSs was also introduced. In this paper, by application of extended modal operators [2] over IFSs, we first show that there are in fact many different ways to construct faithful extensions of Zadeh's inclusion of fuzzy sets, in agreement with various perceptions one may have about IFSs. Secondly, bearing in mind the principle that a two–valued solution to the subsethood assessment problem may be overly restrictive as we may wish to talk about one IFS being a subset of another one up to a certain degree only, we launch a proposal for the definition of graded inclusion measures, nicely encapsulating our findings from the two–valued setting. We briefly relate our work to [6], where the notion of graded inclusion measures over IFSs was obtained by direct generalization of corresponding fuzzy approaches.

2 Preliminaries

A fuzzy set F in a universe X is defined as a mapping from X to $[0, 1]$, such that for each $x \in X$, $F(x)$ (which is sometimes also denoted $\mu_F(x)$, referring explicitly to the membership function μ_F of F) expresses the degree to which x is a member of F. The class of all fuzzy sets in X is denoted $\mathcal{F}(X)$.

[1] gives the following definition of an IFS A in X:

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\} \tag{1}$$

where μ_A and ν_A are called membership and non–membership function of A respectively. They satisfy $\mu_A(x) + \nu_A(x) \leq 1$ for every $x \in X$. The class of all IFSs in X is denoted $\mathcal{IF}(X)$.

An equivalent and more concise way of defining an IFS A in X is as a mapping from X to the set $L^* = \{(x_1, x_2) \in [0,1]^2 \mid x_1 + x_2 \leq 1\}$ shown as a triangle in figure 1. Indeed one may verify that, for $x \in X$, $A(x) = (\mu_A(x), \nu_A(x)) \in L^*$.

Equipping L^* with an ordering $\leq_{L^*}$ defined as $(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$, $(L^*, \leq_{L^*})$ assumes the structure of a complete, bounded lattice with greatest element $1_{L^*} = (1, 0)$ and smallest element $0_{L^*} = (0, 1)$. The sup and inf operations on this lattice are derived from $\leq_{L^*}$ as:

$$\sup((x_1, y_1), (x_2, y_2)) = (\max(x_1, x_2), \min(y_1, y_2)) \tag{2}$$

$$\inf((x_1, y_1), (x_2, y_2)) = (\min(x_1, x_2), \max(y_1, y_2)) \tag{3}$$

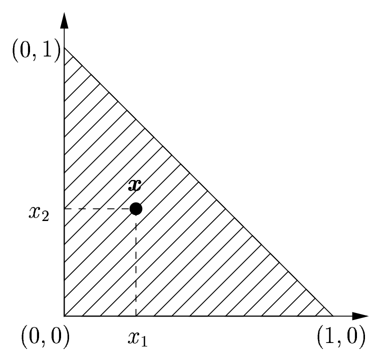

Figure 1: A graphical representation of L^*

Thus, IFSs are a special case of L–fuzzy sets in the sense of Goguen [12], with $L = L^*$.

The (standard) intersection, union and complement of A and $B \in \mathcal{IF}(X)$ are defined by, for $x \in X$, $A \cap B(x) = \inf(A(x), B(x))$, $A \cup B(x) = \sup(A(x), B(x))$, $co(A)(x) = (\nu_A(x), \mu_A(x))$. Standard subsethood for IFSs A and B in X is defined by:

$$\begin{aligned} A \subseteq B &\iff (\forall x \in X)(\mu_A(x) \leq \mu_B(x) \text{ and } \nu_B(x) \geq \nu_A(x)) \\ &\iff (\forall x \in X)(A(x) \leq_{L^*} B(x)) \end{aligned} \tag{4}$$

This definition is faithful, i.e. it extends Zadeh's traditional notion of inclusion for fuzzy sets.

The necessity $\Box A$ and possibility $\Diamond A$ of an IFS A in X are fuzzy sets in X defined by, for $x \in X$, $(\Box A)(x) = \mu_A(x)$ and $(\Diamond A)(x) = 1 - \nu_A(x)$. In [2], a class of generalized modal operators D_α, $\alpha \in [0, 1]$, transforming IFSs to fuzzy sets, was defined by

$$D_\alpha(A)(x) = \mu_A(x) + \alpha\pi_A(x) \tag{5}$$

It is immediately clear that $\Box = D_0$ and $\Diamond = D_1$. The (elementwise) effect of these operators is shown graphically in figure 2.

Intuitively, α can be seen as a parameter of optimism: the operator D_α actually distributes the indeterminacy, captured by π_A, among the membership and non–membership functions of A, in proportions determined by α. The voting example from the introduction may help to visualize this process: if the persons who originally abstained from casting their vote, were somehow *forced* to express a preference in favour or against the proposal, what may we expect their opinion will be like? The necessity of A represents the one extreme situation where pessimistically all indeterminacy is attributed to the non–membership component (everyone not explicitly in favour of the proposal is against it), while in the other limit case of possibility, we (optimistically) conjecture that the indeterminacy, once resolved, will move entirely to the membership component (all not explicitly against the proposal are in favour).

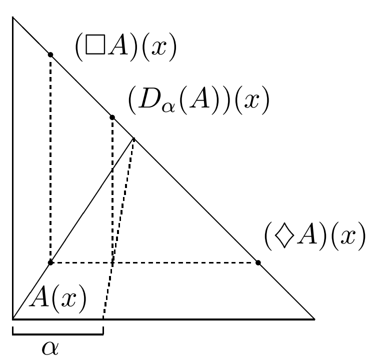

Figure 2: A graphical representation of $\Box A$, $\Diamond A$ and $D_\alpha A$

3 A New Class of Inclusion Indicators

Definition 1 **(α–inclusion)** *Let A, B be IFSs in X, $\alpha \in [0,1]$. We say that A is α–included into B, denoted $A \subseteq_\alpha B$ if and only if $D_\alpha(A) \subseteq D_{1-\alpha}(B)$. Equivalently,*

$$A \subseteq_\alpha B \iff (\forall x \in X)(\mu_A(x) + \alpha\pi_A(x) \leq \mu_B(x) + (1-\alpha)\pi_B(x))$$

As with the extended modal operators, these newly defined operators represent a range of possible attitudes between total pessimism and total optimism in resolving the indeterminacy: in the one limit case of $\subseteq_1$ all the elements are treated as belonging maximally (i.e. as far as the indeterminacy allows to stretch) to A and minimally to B, while $\subseteq_0$ represents the dual optimistic case. We proceed to study some properties of this class of two–valued inclusion indicators and compare them with Atanassov's definition (4).

Theorem 1 *Let A, B be IFSs in X. Then the following statements hold:*

1. *If A and B are fuzzy sets, then $(\forall\alpha \in [0,1])(A \subseteq_\alpha B \iff A \subseteq B)$ ($\subseteq_\alpha$ is a faithful extension of inclusion for fuzzy sets)*
2. $(\forall\alpha_1, \alpha_2 \in [0,1])((A \subseteq_{\alpha_1} B$ *and* $\alpha_1 \geq \alpha_2) \Rightarrow A \subseteq_{\alpha_2} B)$
3. $A \subseteq_1 B \Rightarrow A \subseteq B$
   $A \subseteq B \Rightarrow A \subseteq_{0.5} B$

Proof:

1. When A and B are fuzzy, $\pi_A(x) = \pi_B(x) = 0$ holds for every $x \in X$, so the claim obviously holds.
2. Obvious.
3. First, assume $A \subseteq_1 B$ and let $x \in X$. Then $\mu_A(x) + \pi_A(x) \leq \mu_B(x)$, hence $\mu_A(x) \leq \mu_B(x)$. Moreover, since $\mu_A(x) + \pi_A(x) = 1 - \nu_A(x)$ and $\mu_B(x) \leq 1 - \nu_B(x)$, we also have $\nu_A(x) \geq \nu_B(x)$, so $A \subseteq B$.

   Next, note that $A \subseteq_{0.5} B$ can be rewritten as:

   $$\begin{aligned} A \subseteq_{0.5} B &\iff (\forall x \in X)(\mu_A(x) + 0.5\pi_A(x) \leq \mu_B(x) + 0.5\pi_B(x)) \\ &\iff (\forall x \in X)(\mu_A(x) - \nu_A(x) \leq \mu_B(x) - \nu_B(x)) \end{aligned}$$

   The latter formula clearly holds given $A \subseteq B$. □

On the other hand, from $A \subseteq_\alpha B$, $\alpha \in [0,1[$ in general does not follow $A \subseteq B$. In particular, choose IFSs A and B in $X = \{x\}$ such that $\mu_A(x) = \mu_B(x) + \frac{1-\alpha}{2}\pi_B(x)$, $\pi_A(x) = 0$ and $\pi_B(x) > 0$ (A is fuzzy, B is not). Then clearly $\mu_A(x) + \alpha\pi_A(x) \leq \mu_B(x) + (1-\alpha)\pi_B(x)$, so $A \subseteq_\alpha B$ but $\mu_A(x) > \mu_B(x)$ since $\alpha < 1$, so $A \not\subseteq B$.

4 Graded Inclusion Indicators

Just like definition (4), the inclusion indicators that we introduced in the last section are two–valued: either A is a subset of B, or it isn't. While in many theoretical and practical settings such a characterization is quite sufficient, it could be argued that it is overly restrictive; hence it may be useful to relax the rigid two–valued definitions and allow that A can be a subset of B up to a certain extent. This idea of graded inclusion indicators is relevant for instance in the definition of similarity and non–probabilistic entropy measures, and in approximate reasoning [5].

We can draw much inspiration from fuzzy set theory in this respect, where this problem has already been dealt with extensively. Many researchers [4, 7, 11, 13, 14, 15, 16] have proposed concrete operators Inc that take a couple of fuzzy sets (A, B) in the same universe X as their input and return a value $Inc(A, B)$ in $[0, 1]$ indicating the degree of subsethood of A to B.

A reasonable solution to the subsethood determination problem for IFSs A and B in X seems the following: by applying the extended modal operators from (5) to A and B we obtain couples $(D_\alpha(A), D_\beta(B))$ for every $\alpha, \beta \in [0, 1]$. Let Inc be a fuzzy inclusion measure that satisfies the following basic monotonicity conditions (F, F_1, F_2, G, G_1 and G_2 are all fuzzy sets in the same universe X):

$$F_1 \subseteq F_2 \;\Rightarrow\; Inc(F_1, G) \geq Inc(F_2, G) \tag{6}$$

$$G_1 \subseteq G_2 \;\Rightarrow\; Inc(F, G_1) \leq Inc(F, G_2) \tag{7}$$

Clearly,

$$Inc(\Diamond A, \Box B) \leq Inc(D_\alpha(A), D_\beta(B)) \leq Inc(\Box A, \Diamond B) \tag{8}$$

holds for every choice of α and β in [0,1]. So, a convenient way for aggregating all the available information on A and B (in particular, on all the "fuzzy" situations that may emerge by resolving their indeterminacy) is to define $\mathcal{INC}(A, B) = [Inc(\Diamond A, \Box B), Inc(\Box A, \Diamond B)]$. Since the latter is an interval in $[0,1]$, we may equivalently (see e.g. [9]) write

$$\mathcal{INC}(A,B) = (Inc(\Diamond A, \Box B), 1 - Inc(\Box A, \Diamond B)) \tag{9}$$

where $\mathcal{INC}$ is thus regarded as an $\mathcal{IF}(X) \times \mathcal{IF}(X) \to L^*$ mapping. This transformation is particularly interesting as it allows us to compare (9) with an earlier proposal from [6] of L^*–valued inclusion indicators over IFSs.

One of the ideas advanced in [6] was to consider a direct generalization of certain fuzzy inclusion measures to IFS theory. It is called "direct" in a sense that operations on the unit interval featuring in the definitions of suitable fuzzy inclusion measures are replaced by some of their extensions to L^*. For hints on how to obtain a suitable fuzzy inclusion measure, we refer to e.g. [6, 7] and [16]. Let us suffice to mention that a good candidate in many situations is the Łukasiewicz inclusion measure, defined by, for F and G fuzzy sets in X:

$$Inc_L(F,G) = \inf_{x \in X} \min(1, 1 - F(x) + G(x)) \tag{10}$$

In [6], the following two alternative generalizations of (10) to IFSs A and B in X were proposed:

$$\begin{aligned} \mathcal{INC}_1(A,B) &= \Big(\inf_{x \in X} \min(1, \mu_B(x) + 1 - \mu_A(x), \nu_A(x) + 1 - \nu_B(x)), \\ &\quad \sup_{x \in X} \max(0, \mu_A(x) + \nu_B(x) - 1) \Big) \qquad (11) \\ \mathcal{INC}_2(A,B) &= \Big(\inf_{x \in X} \min(1, \nu_A(x) + \mu_B(x)), \\ &\quad \sup_{x \in X} \max(0, \mu_A(x) + \nu_B(x) - 1) \Big) \qquad (12) \end{aligned}$$

with a marked preference of (11) over (12) as the latter exhibits the following "unusual" behaviour:

$$A \subseteq B \not\Rightarrow \mathcal{INC}_2(A,B) = 1_{L^*} \tag{13}$$

So, it is not an extension of standard inclusion over IFSs (but it does faithfully generalize both Inc_L and Zadeh's inclusion over fuzzy sets). As we have seen in the previous section that formula (4) is not the only reasonable option for defining two–valued subsethood over IFSs, (13) seems at once a much less compelling argument to dismiss $\mathcal{INC}_2$. The following theorem sheds some more light on the matter.

Theorem 2 *When $\mathcal{INC}$ is defined by means of the fuzzy inclusion measure Inc_L, it coincides totally with $\mathcal{INC}_2$, i.e. for arbitrary IFSs A and B in X, $\mathcal{INC}(A,B) = \mathcal{INC}_2(A,B)$.*

Proof: we may rewrite $\mathcal{INC}(A,B)$ as follows:

$$
\begin{aligned}
\mathcal{INC}(A,B) &= (Inc_L(\Diamond A, \Box B), 1 - Inc_L(\Box A, \Diamond B)) \\
&= \Big(\inf_{x\in X} \min(1, 1 - (\Diamond A)(x) + (\Box B)(x)), \\
&\quad 1 - \inf_{x\in X} \min(1, 1 - (\Box A)(x) + (\Diamond B)(x))\Big) \\
&= \Big(\inf_{x\in X} \min(1, 1 - (1 - \nu_A(x)) + \mu_B(x)), \\
&\quad \sup_{x\in X} 1 - \min(1, 1 - \mu_A(x) + 1 - \nu_B(x))\Big) \\
&= \Big(\inf_{x\in X} \min(1, \nu_A(x) + \mu_B(x)), \\
&\quad \sup_{x\in X} \max(0, \mu_A(x) + \nu_B(x) - 1)\Big) \\
&= \mathcal{INC}_2(A,B)
\end{aligned}
$$

□

Theorem 3 *For arbitrary IFSs A and B in X,*

$$A \subseteq_1 B \Rightarrow \mathcal{INC}_2(A,B) = 1_{L^*} \tag{14}$$

Proof: from $A \subseteq_1 B$ follows, for u in U, $1 - \nu_A(u) \leq \mu_B(u)$, or $\nu_A(u) + \mu_B(u) \geq 1$, so $\min(1, \nu_A(u) + \mu_B(u)) = 1$, and hence $\mathcal{INC}_2(A,B) = 1_{L^*}$. □

So, $\mathcal{INC}_2$ extends the strong inclusion indicator $\subseteq_1$ rather than the standard inclusion indicator, and thus stands as a reasonable alternative to $\mathcal{INC}_1$.

5 Conclusion

In this paper, by introducing a class $\subseteq_\alpha$ of two–valued inclusion indicators whose semantics were motivated in terms of the extended modal operators and the indeterminacy of IFSs, we have shown that many reasonable alternatives exist to the original definition of inclusion over IFSs. This observation has also important repercussions for the definition of graded inclusion measures over IFSs; in particular, we could establish links between earlier results on direct generalizations of fuzzy inclusion measures, and a novel approach based on the newly defined class.

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