RELATING INTUITIONISTIC FUZZY SETS AND INTERVAL-VALUED FUZZY SETS THROUGH BILATTICES

O. ARIELI

Department of Computer Science,
The Academic College of Tel-Aviv,
Antokolski 4, Tel-Aviv 61161, Israel
E-mail: oarieli@mta.ac.il

C. CORNELIS,* G. DESCHRIJVER AND E. E. KERRE

Department of Applied Mathematics and Computer Science,
Ghent University,
Krijgslaan 281 (S9), B-9000 Gent, Belgium
E-mail: {chris.cornelis,glad.deschrijver,etienne.kerre}@ugent.be

In this paper, we show that bilattices are robust mathematical structures that provide a natural accommodation to, and bridge between, intuitionistic fuzzy sets and interval-valued fuzzy sets. In this way, we resolve the controversy surrounding the formal equivalence of these two models, and open up the path for a new tradition for representing positive and negative information in fuzzy set theory.

1. Motivation

Bilattices are algebraic structures that were introduced by Ginsberg[14], and further examined by Fitting[12,13] and others, e.g.[2], as a general framework for many applications in computer science. In this paper, we show that these structures can also elegantly and naturally accomodate intuitionistic fuzzy sets (IFSs) and interval-valued fuzzy sets (IVFSs), which are two frequently encountered and syntactically equivalent generalizations of Zadeh's fuzzy sets. In particular, and more generally than in previous works, we demonstrate that Atanassov's decision to restrict the evaluation set for L-intuitionistic fuzzy sets to *consistent* couples of the "square" $\mathcal{L}^2$ forces the resulting structure to coincide with the "triangle" $\mathcal{I}(\mathcal{L})$. This insight provides a convenient stepping stone towards more general and expressive

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models for the representation and processing of positive and negative imprecise information.

2. Preliminaries: IFSs, IVFSs, and Bilattices

2.1. *Intuitionistic fuzzy sets (IFSs)*

A *fuzzy set*[17] is a nebular collection of elements from a universe U, described by a *membership function* $\mu : U \to [0,1]$. An *intuitionistic fuzzy set*[3] (IFS, for short) is a nebular collection of elements from a universe U, described by a *pair* of functions (μ, ν), each one maps elements from U to the unit interval $[0,1]$, such that for every u in U, $\mu(u) + \nu(u) \leq 1$. Intuitively, μ is a membership function and ν is a non-membership function. These two functions are not necessarily each other's complement (an assumption which is implicit in Zadeh's fuzzy set theory), i.e., the amount of the 'missing information', $1 - \mu(u) - \nu(u)$, may be strictly positive.

Given a complete lattice $\mathcal{L} = (L, \leq)$, Goguen[15] introduced the concept of *L–fuzzy sets* as a mapping $\mu : U \to L$. Intuitionistic fuzzy sets can be interpreted as a particular kind of L–fuzzy sets, where the corresponding complete lattice is the following[10]:

Definition 2.1. Define: $\mathcal{L}^* = (L^*, \leq_{L^*})$, where $L^* = \{(x_1, x_2) \mid (x_1, x_2) \in [0,1] \times [0,1] \text{ and } x_1 + x_2 \leq 1\}$, and $(x_1, x_2) \leq_{L^*} (y_1, y_2)$ iff $x_1 \leq y_1$ and $x_2 \geq y_2$.

Atanassov and Stoeva[4] introduced the following generalization of the IFS construct, called an *intuitionistic L–fuzzy set* (ILFS).

Definition 2.2. Let $(L, \leq_L)$ be a complete lattice with an involution operation[a] $\mathcal{N}$ and a non–empty set U called universe. An *intuitionistic L–fuzzy set* in U is a mapping $g : U \to L \times L$, such that if $g(u) = (x_1, x_2)$ then $x_1 \leq_L \mathcal{N}(x_2)$, for all u in U.

2.2. *Interval-valued fuzzy sets (IVFSs)*

In interval-valued fuzzy sets the membership degrees are represented by intervals in $[0,1]$ that approximate the correct (but unknown) membership degree. Another justification for this approach is that, in reality, intervals of values better reflect experts' opinions than exact numbers.

An IVFS can be seen as an L^I–fuzzy set, where the corresponding lattice is given by the following definition.

[a] I.e., for every x, y in L, $\mathcal{N}(\mathcal{N}(x)) = x$, and if $x \leq_L y$ then $\mathcal{N}(x) \geq_L \mathcal{N}(y)$.

Definition 2.3. Define: $\mathcal{L}^I = (L^I, \leq_{L^I})$, where $L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0,1] \times [0,1]$ and $x_1 \leq x_2\}$, and $[x_1, x_2] \leq_{L^I} [y_1, y_2]$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$.

Atanassov and Gargov[5] defined another generalization of the IFS construct, called *interval–valued intuitionistic fuzzy set* (IVIFS), which deviates from the line of thinking of ILFS, and which is more related to the intuition behind IVFSs.

Definition 2.4. An *interval-valued intuitionistic fuzzy set*[5] in a universe U is a mapping $g : U \to (L^I)^2$, such that $g(u) = ([x_1^l, x_1^h], [x_2^l, x_2^h])$ and $x_1^h + x_2^h \leq 1$, for all u in U.

Indeed, applying Definition 2.2 to $(L^I, \leq_{L^I})$, where the involution $\mathcal{N}$ on the lattice $(L^I, \leq_{L^I})$ is defined by $\mathcal{N}([x_1, x_2]) = [1 - x_2, 1 - x_1]$, gives the alternative condition $[x_1^l, x_1^h] \leq_{L^I} [1 - x_2^h, 1 - x_2^l]$.

2.3. *Bilattices*

As noted above, bilattices are used here for relating IFSs and IVFSs. First, we recall some basic definitions and notions that are related to these structures.

Definition 2.5. A *pre-bilattice*[12] is a structure $\mathcal{B} = (B, \leq_t, \leq_k)$, such that B is a nonempty set containing at least two elements, and $(B, \leq_t)$, $(B, \leq_k)$ are complete lattices.

Definition 2.6. A *bilattice*[14] is a structure[b] $\mathcal{B} = (B, \leq_t, \leq_k, \neg)$, such that $(B, \leq_t, \leq_k)$ is a pre-bilattice, and $\neg$ is a unary operation on B that has the following properties: for every x, y in B,

(1) if $x \leq_t y$ then $\neg x \geq_t \neg y$, (2) if $x \leq_k y$ then $\neg x \leq_k \neg y$, (3) $\neg\neg x = x$.

The original motivation of Ginsberg[14] for using bilattices was to provide a uniform approach for a diversity of applications in AI. In particular, he considered first-order theories and their consequences, truth maintenance systems, and default reasoning. Later, it was shown that bilattices are

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[b]Note that Definition 2.6 is not the same as the one in [7], but rather corresponds to Ginsberg's original definition of bilattices. In terms of Fitting, the definition above describes a *pre-bilattice with a negation*, while the structures considered in [7] are *pre-bilattices that are interlaced.* As a bilattice may not be interlaced on one hand, and it may not be possible to define a negation operator for a given interlaced pre-bilattice on the other hand, the present definition of bilattices is incomparable with that of [7]. This will not be an obstacle in what follows, though.

useful for giving semantics to logic programs[11,12] and that they provide an intuitive semantics to consequence relations for reasoning with uncertainty[1].

Following the conventional notations in the literature, we shall denote by $\wedge$ (by $\vee$) $\leq_t$-meet ($\leq_t$-join), and by $\otimes$ (by $\oplus$) $\leq_k$-meet ($\leq_k$-join) of a bilattice $\mathcal{B}$; f and t will denote the extreme elements of $(B, \leq_t)$, and $\bot$, $\top$ will denote the extreme elements of $(B, \leq_k)$.

In some bilattices a dual negation operator, called *conflation*[12] $(-)$, is definable. It is an involution of $(B, \leq_k)$ and order preserving of $(B, \leq_t)$:

(1) if $x \leq_k y$ then $-x \geq_k -y$, (2) if $x \leq_t y$ then $-x \leq_t -y$, (3) $--x = x$.

Proposition 2.1. *Let $\mathcal{B} = (B, \leq_t, \leq_k, \neg)$ be a bilattice. Then:*

a) [14] $\neg f = t$, $\neg t = f$, $\neg\bot = \bot$, $\neg\top = \top$. *Also, for every x, y in B,* $\neg(x \wedge y) = \neg x \vee \neg y$, $\neg(x \vee y) = \neg x \wedge \neg y$, $\neg(x \otimes y) = \neg x \otimes \neg y$, $\neg(x \oplus y) = \neg x \oplus \neg y$.

b) [12] *If $\mathcal{B}$ has a conflation, then* $-f = f$, $-t = t$, $-\bot = \top$, $-\top = \bot$. *For every x, y in B,* $-(x \wedge y) = -x \wedge -y$, $-(x \vee y) = -x \vee -y$, $-(x \otimes y) = -x \oplus -y$, *and* $-(x \oplus y) = -x \otimes -y$.

3. Squares and Triangles

3.1. *Squares*

Definition 3.1. [14] Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice. The structure $\mathcal{L}^2 = (L \times L, \leq_t, \leq_k, \neg)$ is defined as follows: (1) $\neg(x_1, x_2) = (x_2, x_1)$,
(2) $(x_1, x_2) \leq_t (y_1, y_2)$ iff $x_1 \leq_L y_1$ and $x_2 \geq_L y_2$,
(3) $(x_1, x_2) \leq_k (y_1, y_2)$ iff $x_1 \leq_L y_1$ and $x_2 \leq_L y_2$.

In what follows we refer to $\mathcal{L}^2$ as a *square*. A pair $(x_1, x_2) \in \mathcal{L}^2$ may intuitively be understood so that x_1 represents the amount of belief *for* some assertion, and x_2 is the amount of belief *against* it. This is clearly the same idea as that of Atanassov[3], discussed in Section 2.1, of splitting a belief about the membership of an element u to two components $(\mu(u), \nu(u))$. As we shall show, the similarity does not remain only on this intuitive level.

Proposition 3.1. *Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice with a join $\sqcap_L$ and a meet $\sqcup_L$. Then:*

a) [14] *$\mathcal{L}^2$ is a bilattice, in which* $\bot_{\mathcal{L}^2} = (\inf(L), \inf(L))$, $\top_{\mathcal{L}^2} = (\sup(L), \sup(L))$, $t_{\mathcal{L}^2} = (\sup(L), \inf(L))$, $f_{\mathcal{L}^2} = (\inf(L), \sup(L))$.

The basic operations in $\mathcal{L}^2$ are defined as follows: $\neg(x_1, x_2) = (x_2, x_1)$, $(x_1, x_2) \vee (y_1, y_2) = (x_1 \sqcup_L y_1, x_2 \sqcap_L y_2)$, $(x_1, x_2) \wedge (y_1, y_2) = (x_1 \sqcap_L y_1, x_2 \sqcup_L y_2)$,

$(x_1, x_2) \oplus (y_1, y_2) = (x_1 \sqcup_L y_1, x_2 \sqcup_L y_2)$, $(x_1, x_2) \otimes (y_1, y_2) = (x_1 \sqcap_L y_1, x_2 \sqcap_L y_2)$.

b) [11] *Suppose that $\mathcal{L}$ has an involution. Denote by x^- the $\leq_L$-involute of x in L. Then a conflation is defined on $\mathcal{L}^2$ by $-(x_1, x_2) = (x_2^-, x_1^-)$.*

Example 3.1. The square derived from $(\{0, \frac{1}{2}, 1\}, \leq)$ is shown in Figure 1 (left); Belnap's four-valued bilattice[6] is obtained by the square that is derived from $(\{0, 1\}, \leq)$.

Remark that $\mathcal{B}$ endorses the main intuition behind L–IFSs: if the membership degree of an element u is x in L, it is not necessarily the case that the non-membership degree of u is $1 - x$, but it is rather some y in L. Also note that, when $L = [0, 1]$, the structure $\mathcal{B} = \mathcal{L}^2$ simultaneously encompasses the order relations among IVFSs and IFSs: the $\leq_k$–ordering of this bilattice is exactly the same as the partial order of $\mathcal{L}^I$ (Definition 2.3). The $\leq_t$–order of $\mathcal{B}$, on the other hand, corresponds to the partial order of $\mathcal{L}^*$ (Definition 2.1).

3.2. *Triangles*

Definition 3.2. [12] For a complete lattice $\mathcal{L} = (L, \leq_L)$, let $\mathcal{I}(\mathcal{L})$ be a triple $(I(L), \leq_t, \leq_k)$ with a set $I(L)$ of intervals $[x_1, x_2] = \{x \mid x \in L \text{ and } x_1 \leq_L x \leq_L x_2\}$, s.t. (1) $[x_1, x_2] \leq_t [y_1, y_2]$ iff $x_1 \leq_L y_1$ and $x_2 \geq_L y_2$,
(2) $[x_1, x_2] \leq_k [y_1, y_2]$ iff $x_1 \leq_L y_1$ and $x_2 \leq_L y_2$.

An interval $[y_1, y_2]$ in $I(L)$ is $\leq_k$–greater (i.e., more informative) than $[x_1, x_2]$ in $I(L)$ if $[y_1, y_2] \subseteq [x_1, x_2]$; it is $\leq_t$–greater than $[x_1, x_2]$ if $(\forall x \in [x_1, x_2])$ $(\exists y \in [y_1, y_2])$ $(x \leq_L y)$ and $(\forall y \in [y_1, y_2])$ $(\exists x \in [x_1, x_2])$ $(y \geq_L x)$. Note that $\mathcal{I}(\mathcal{L})$ is not closed under $\leq_k$–join, and so it is not a (pre–)bilattice but only a so–called pseudo $\leq_k$–lower pre-bilattice[12]. In what follows we shall call $\mathcal{I}(\mathcal{L})$ a *triangle*. The triangle $\mathcal{I}(\{0, \frac{1}{2}, 1\})$ is shown in Figure 1 (right).

When $\mathcal{L}$ is the unit interval, $\mathcal{I}(\mathcal{L})$ naturally describes membership of IVFSs, and the valuation lattice $\mathcal{L}^I$ is exactly $(I(L), \leq_t)$. Moreover, $\mathcal{I}(\mathcal{L})$ extends $\mathcal{L}^I$ in the sense that it contains the partially ordered set $(I(L), \leq_k)$, that orders the intervals according to their amount of information.

Definition 3.3. [12] Let $\mathcal{B}$ be a bilattice with a conflation. An element x in B is called *exact* if $x = -x$; it is *consistent* if $x \leq_k -x$.

Proposition 3.2. [12] *Let $\mathcal{L}$ be a complete lattice with involution. Then $\mathcal{I}(\mathcal{L})$ is isomorphic to the structure of the consistent elements of $\mathcal{L}^2$.*

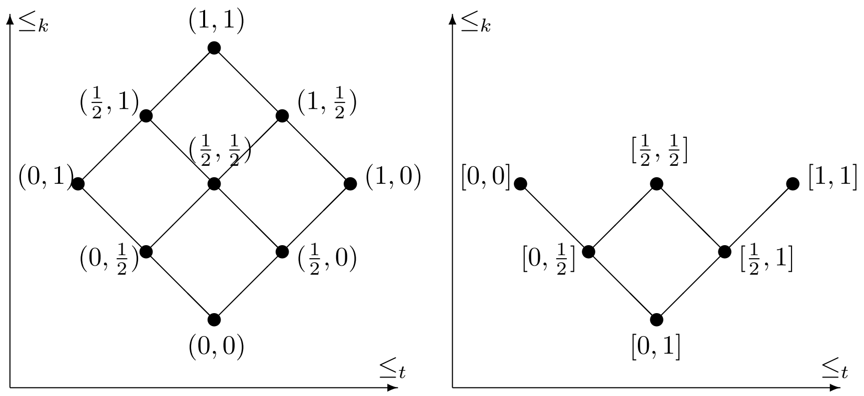

Figure 1. The square $\{0, \frac{1}{2}, 1\}^2$ and the triangle $\mathcal{I}(\{0, \frac{1}{2}, 1\})$

The last proposition allows to put several matters into the right perspective:

(1) The isomorphism $f : \mathcal{I}(\mathcal{L}) \to \mathcal{L}^2$ for the proof of Proposition 3.2 is given by $f([x_1, x_2]) = (x_1, -x_2)$, where $-x_2$ is the involute of x_2 in L. For the unit interval, then, $f([x_1, x_2]) = (x_1, 1-x_2)$, which is the same transformation considered by Cornelis et al.[8] for switching between IVFSs and IFSs. Proposition 3.2 shows that the same transformation is useful not only for $\mathcal{L}^I$ and $\mathcal{L}^*$ (i.e., when the underlying lattice is the unit interval), but for *any* complete lattice with involution.

(2) Proposition 3.2 may also serve as a justification for Atanassov's decision to consider only the "lower triangle" of $[0, 1]^2$ (i.e., the elements in (a, b) in $[0, 1]^2$ s.t. $a + b \leq 1$): these are exactly the consistent elements of $[0, 1]^2$ and so, as Proposition 3.2 above indicates, the lattice $\mathcal{L}^*$ (of the consistent elements in $[0, 1]$) is isomorphic to the lattice $\mathcal{L}^I$ of the $[0, 1]$-interval-valued fuzzy sets. The fact that we consider super-lattices of $\mathcal{L}^*$ (i.e., all the elements in $[0, 1]^2$) allows us to introduce elements such as $(a, b) = (1, 1)$, in which the membership degree (a) and the non-membership degree (b) are both maximal. This means that we have a totally inconsistent belief in this case. As an important aspect of fuzzy logic is reasoning with imprecise and possibly conflicting information, such values should not be ruled out!

(3) Remarkably, Atanassov and Stoeva's ILFS construct[4] (Definition 2.2), is the exact embodiment of Proposition 3.2. It is not difficult to see that the condition on g in Definition 2.2 means that $g(u)$ is a consistent element of $\mathcal{L}^2$, where the conflation is defined as $-(x_1, x_2) = (\mathcal{N}(x_2), \mathcal{N}(x_1))$, for every $(x_1, x_2) \in L^2$.

(4) Pankowska and Wygralak[16] introduced a kind of L–IFSs based on the lattice $([0,1], \leq)$ with an involution operation defined, for any positive real number n, by $\mathcal{N}_n(x) = \sqrt[n]{1 - x^n}$. It is easy to see that when n increases, so does the number of elements (x_1, x_2) in $[0,1]^2$ for which $x_1 \leq \mathcal{N}_n(x_2)$, or equivalently $x_2 \leq \mathcal{N}_n(x_1)$. In fact, if $x_1 \neq 1$, then $\lim_{n \to +\infty} \mathcal{N}_n(x_1) = 1$, but when $x_1 = 1$, always $\mathcal{N}_n(x_1) = 0$. Hence, when n approaches $+\infty$, the set of consistent elements of $[0,1] \times [0,1]$ approaches $[0,1]^2$ *without* the elements $(x_1, 1)$ and $(1, x_2)$, for which $x_1, x_2 > 0$.

4. Concluding Remarks

Bilattices are rich mathematical structures that nicely reflect the intuitions behind IFSs and IVFSs and relate the corresponding theories. The square $\mathcal{L}^2$, together with its 'information order' $\leq_k$, generalizes the [0,1]-interval-valued structure $\mathcal{L}^I$, and at the same time $\mathcal{L}^2$ with its 'truth order' $\leq_t$ extends the [0,1]-intuitionistic fuzzy structure $\mathcal{L}^*$. In particular,

- lattices other than the unit interval fit into Atanassov's framework, and
- inconsistent elements are also allowed (for representing inconsistent beliefs).

These considerations open up new and challenging directions for (L-) fuzzy set theory, in particular, w.r.t. the processing of collections of partial and potentially conflicting positive and negative information items.

References

1. O.Arieli, A.Avron. Reasoning with logical bilattices. *Journal of Logic, Language, and Information* 5(1), pp.25–63, 1996.
2. O.Arieli, A.Avron. Bilattices and paraconsistency. In: *Frontiers of Paraconsistent Logic,* D.Batens, C.Mortensen, G.Priest, and J.Van Bendegem, editors, Studies in Logic and Computation 8, Research Studies Press, pp.11–27, 2000.

3. K.T.Atanassov. Intuitionistic fuzzy sets. *VII ITKR Session* (1983). Report No.1697/84 of the Cental. Sci.–Techn. Library of the Bulg. Acad. of Sci.
4. K.T.Atanassov, S.Stoeva. Intuitionistic L–fuzzy sets, *Cybernetics and Systems Research* 2, pp.539–540, 1984.
5. K.T.Atanassov, G.Gargov. Interval–valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems* 31(3), pp.343–349, 1989.
6. N.D.Belnap. A useful four-valued logic. *Modern Uses of Multiple-Valued Logic* Reidel Publishing Company, pp.7–37, 1977.
7. C.Cornelis, K.T.Atanassov, E.Kerre. Intuitionistic fuzzy sets and interval-valued fuzzy sets: A critical comparison. *Proc. EUSFLAT'03*, pp.159–163, 2003.
8. C.Cornelis, G.Deschrijver, E.Kerre. Implication in intuitionistic and interval-valued fuzzy set theory. *J. Approximate Reasoning* 35(1), pp.55–95, 2004.
9. G.Deschrijver, C.Cornelis, E.Kerre. Square and triangle: a comparison, Notes on Intuitionistic Fuzzy Sets 9(3), pp. 11–21, 2003.
10. G.Deschrijver, E.Kerre. On the relationship between some extensions of fuzzy set theory. *Fuzzy Sets and Systems* 133(2), pp.227–235, 2003.
11. M.Fitting. Bilattices in logic programming. *Proc. 20th Int. Symposium on Multiple-Valued Logic*, IEEE Press, pp.238–246, 1990.
12. M.Fitting. Kleene's logic, generalized. *J. Logic and Comput.* 1, 797–810, 1992.
13. M.Fitting. Kleene's three-valued logics and their children. *Fundamenta Informaticae* 20 (1994), 113-131.
14. M.L.Ginsberg. Multi-valued logics: A uniform approach to reasoning in artificial intelligence, *Computer Intelligence* 4, pp.256–316, 1988.
15. J.Goguen. L–fuzzy sets. *Journal Math. Anal. Appl.* 18, pp.145–174, 1967.
16. A.Pankowska, M.Wygralak. Intuitionistic fuzzy sets – An alternative look. *Proc. EUSFLAT'03*, 2003.
17. L.A.Zadeh. Fuzzy sets. *Information and Control* 8, pp.338–353, 1965.