LINGUISTIC HEDGES IN AN INTUITIONISTIC FUZZY SETTING

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ABSTRACT

Slowly but surely, intuitionistic fuzzy sets are giving away their secrets. By tracing them back to the underlying algebraic structure that they are defined on (a complete lattice), they can be embedded in the well–known class of L–fuzzy sets, whose formal treatment allows the definition and study of order–theoretic concepts such as triangular norms and conorms, negators and implicators, as well as the development of more complex operations such as direct and superdirect image,... In this paper we use the latter for the representation of linguistic hedges. We study their behaviour w.r.t. hesitation, and we examine how in this framework modification of an intuitionistic fuzzy set can be constructed from separate modification of its membership and non-membership function.

1. INTRODUCTION

Since its introduction in the sixties, fuzzy set theory [14] has rapidly acquired an immense popularity as a formalism for the representation of vague linguistic information. Over the years many researchers have studied the automatic computation of membership functions for modified linguistic terms (such as very cool) from those of atomic ones (such as cool). In this paper, we consider this topic for intuitionistic fuzzy sets (IFSs for short), a variation on the original theme of a fuzzy set, which were introduced by Atanassov [1] and which are currently generating a great deal of interest.

IFS theory basically defies the claim that from the fact that an element u "belongs" to a given degree (say $\mu_A(u)$) to a fuzzy set A, naturally follows that u should "not belong" to A to the extent $1 - \mu_A(u)$, an assertion implicit in the concept of a fuzzy set. On the contrary, IFSs assign to each element u of the universe both a degree of membership $\mu_A(u)$ and one of non–membership $\nu_A(u)$ such that $\mu_A(u) + \nu_A(u) \leq 1$, thus relaxing the enforced duality $\nu_A(u) = 1 - \mu_A(u)$ from fuzzy set theory. Obviously, when $\mu_A(u) + \nu_A(u) = 1$ for all elements of the universe, the traditional fuzzy set concept is recovered.

Wang and He [13], and later also Deschrijver and Kerre [7], noticed that IFSs can be considered as special instances of Goguen's L–fuzzy sets [9], so every concept definable for L–fuzzy sets is also available to IFS theory. In this spirit, in [6, 8] suitable definitions and representation theorems for the most important intuitionistic fuzzy connectives have been derived; negators, triangular norms and conorms, and implicators can be used to model the elementary set operations of complementation, intersection and union, as well as the logical operations of negation, conjunction, disjunction and implication.

Going one step further, we can obtain representations of linguistic hedges (also called linguistic modifiers) such as more or less and very for IFSs using the context based relational approach of De Cock and Kerre [5] for terms modeled by L–fuzzy sets. In this framework, more or less is modeled by means of the direct image based on some triangular norm, while very can be represented by the superdirect image based on some implicator. The approach boasts in general a lot of nice properties as well as many practical and intuitive advantages over "traditional" modifiers such as powering [14] and shifting hedges [11].

Another way to define linguistic modifiers for IFSs stems from the "divide–and–conquer" rationale: we impose suitable modifications on the membership and non–membership functions, and ensure that the resulting construct is still an IFS, thus effectively breaking up our original problem into simpler, better–understood tasks.

This paper is structured as follows: after the neces-

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sary preliminaries on intuitionistic fuzzy sets (Section 2), we describe the context based approach for L-fuzzy sets in general, and study a first example in the specific intuitionistic fuzzy setting (Section 3). Inspired by a comparison with an existing but semantically less interesting approach, in Section 4 we start a search for divide-and-conquer opportunities of the context based model.

2. PRELIMINARIES

Definition 1 (Connectives in a lattice) *Let $(L, \leq_L)$ be a complete lattice. A negator on L is any decreasing $L \to L$ mapping $\mathcal{N}$ satisfying $\mathcal{N}(0_L) = 1_L$. It is called involutive if $\mathcal{N}(\mathcal{N}(x)) = x$ for all x in L. A triangular norm (t-norm for short) $\mathcal{T}$ on L is any increasing, commutative and associative $L^2 \to L$ mapping satisfying $\mathcal{T}(1_L, x) = x$, for all x in L. A triangular conorm (t-conorm for short) $\mathcal{S}$ on L is any increasing, commutative and associative $L^2 \to L$ mapping satisfying $\mathcal{S}(0_L, x) = x$, for all x in L. An implicator on L is any $L^2 \to L$–mapping $\mathcal{I}$ satisfying $\mathcal{I}(0_L, 0_L) = 1_L, \mathcal{I}(1_L, 0_L) = 0_L, \mathcal{I}(0_L, 1_L) = 1_L, \mathcal{I}(1_L, 1_L) = 1_L$. Moreover we require $\mathcal{I}$ to be decreasing in its first, and increasing in its second component.*

If $\mathcal{S}$ and $\mathcal{N}$ are respectively a t-conorm and a negator on L, then it is well known that the mapping $\mathcal{I}$ defined by $\mathcal{I}(x, y) = \mathcal{S}(\mathcal{N}(x), y)$ is an implicator on L, usually called S-implicator.

Definition 2 (L-fuzzy set) *Let $(L, \leq_L)$ be a lattice. An L-fuzzy set A in a universe U is a mapping from U to L. The L-fuzzy set A is said to be included in the L-fuzzy set B, usually denoted by $A \subseteq B$, if $A(u) \leq_L B(u)$ for all u in U. For $\mathcal{N}$ a negator on L, the $\mathcal{N}$-complement of A is the L-fuzzy set defined by $co_{\mathcal{N}}A(u) = \mathcal{N}(A(u))$ for all u in U.*

As mentioned in the introduction, intuitionistic fuzzy sets can be defined either in terms of a membership and non–membership function, or as particular instances of L–fuzzy sets. In practice, it is instructive to always have both alternatives in mind, a decision reflected in the following definition.

Definition 3 (Intuitionistic fuzzy set) *Let $(L^*, \leq_{L^*})$ be the complete, bounded lattice defined by [6]:*

$$L^* = \{(x_1, x_2) \in [0,1]^2 \mid x_1 + x_2 \leq 1\}$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2$$

The units of this lattice are denoted $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$. For each element $x \in L^*$, by x_1 and x_2 we denote its first and second component, respectively.

An intuitionistic fuzzy set A in a universe U is a mapping from U to L^*. For every $u \in U$, the value $\mu_A(u) = (A(u))_1$ is called the membership degree of u to A; the value $\nu_A(u) = (A(u))_2$ is called the non–membership degree of u to A; and the value $\pi_A(u) = 1 - \mu_A(u) - \nu_A(u)$ is called the hesitation degree of u to A.

The class of intuitionistic fuzzy sets in U is denoted by $\mathcal{IFS}(U)$. An element of $\mathcal{IFS}(U \times U)$ is called an intuitionistic fuzzy relation in U. The definition of intuitionistic fuzzy t–norms, t–conorms and implicators is immediate by replacing L by L^* in Definition 1. The class of intuitionistic fuzzy t–norms (t–conorms) is then subdivided in the so–called t–representable and non t–representable instances.

Definition 4 (t–representability) *[6] A t–norm $\mathcal{T}$ on L^* (resp. t–conorm $\mathcal{S}$) is called t–representable if there exists a t–norm T and a t–conorm S on $[0, 1]$ (resp. a t–conorm S' and a t–norm T' on $[0, 1]$) such that, for $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,*

$$\mathcal{T}(x, y) = (T(x_1, y_1), S(x_2, y_2))$$
$$\mathcal{S}(x, y) = (S'(x_1, y_1), T'(x_2, y_2))$$

T and S (resp. S' and T') are called the representants of $\mathcal{T}$ (resp. $\mathcal{S}$).

In [6] it is shown that for instance the product and the probabilistic sum on $[0, 1]$ constitute the t-norm $\mathcal{T}_{\mathrm{P}}$ on L^* defined as $\mathcal{T}_{\mathrm{P}}(x, y) = (x_1 \cdot y_1, x_2 + y_2 - x_2 \cdot y_2)$ for all x and y in L^*. Likewise the Łukasiewicz t-norm and t-conorm on $[0, 1]$ give rise to the t-norm $\mathcal{T}_{\mathrm{W}}$ on L^* defined as $\mathcal{T}_{\mathrm{W}}(x, y) = (\max(x_1 + y_1 - 1, 0), \min(x_2 + y_2, 1))$.

Finally, denoting the first projection mapping on L^* by pr_1, we recall from [6] that the $[0, 1] - [0, 1]$ mapping N defined by $N(a) = pr_1\mathcal{N}(a, 1 - a)$ for all a in $[0, 1]$ is an involutive negator on $[0, 1]$, as soon as $\mathcal{N}$ is an involutive negator on L^*. N is called the fuzzy negator induced by $\mathcal{N}$. Furthermore $\mathcal{N}(x_1, x_2) = (N(1 - x_2), 1 - N(x_1))$, for all x in L^*.

3. A CONTEXT BASED MODEL

Whether we are working with fuzzy sets, IFSs or L–fuzzy sets in general, establishing a concrete mathematical model for a given linguistic expression is typically one of the most difficult tasks when developing an application. Therefore it is very useful to have standard representations of linguistic modifiers such as very and more or less at hand, since they allow for the automatic construction of representations for the modified terms from representations of the original terms.

The first proposal in this direction was made by Zadeh [14] who suggested to transform the membership function of a fuzzy set A into membership functions for very A and more or less A in the following way

$$\begin{gathered}\textsf{very } A(v) = A(v)^2\\ \textsf{more or less } A(v) = A(v)^{\frac{1}{2}}\end{gathered}$$

for all v in U. One can easily verify that the following natural condition, called semantical entailment, [11] is respected:

$$\textsf{very } A \subseteq A \subseteq \textsf{more or less } A$$

Today however it is well–known that these representations have the significant shortcoming of keeping the kernel and the support, which are defined as

$$\begin{aligned}\ker A &= \{u|u \in U \wedge A(u) = 1\}\\ \text{supp } A &= \{u|u \in U \wedge A(u) > 0\}\end{aligned}$$

As a consequence they do not make any distinction between e.g. being old to degree 1 and being very old to degree 1, while intuition might dictate to call a man of 85 old to degree 1 but very old only to a lower degree. Many representations developed in the same period are afflicted with these and other disadvantages on the level of intuition as well as on the level of applicability (we refer to [10] for an overview), in our opinion due to the fact that these operators are only technical tools, lacking inherent meaning. In fact it wasn't until the second half of the 1990's that new models with a clear semantics started to surface, such as the horizon approach [12] and the context (or fuzzy relational) based approach [4]. The latter can be elegantly generalized to L-fuzzy sets [5] which accounts for its strength.

A characteristic of the "traditional" approaches is that they do not really look at the context: when computing the degree to which v is very A, Zadeh's representation for instance only looks at the degree to which v is A. It completely ignores all the other objects of the universe and their degree of belonging to A. In the context based approach the objects in the context of v are taken into account as well. This context is defined as the set of objects that are related to v by some relation R that models approximate equality. Specifically it is the L-fuzzy set Rv defined by $Rv(u) = R(u, v)$, for all u in U.

One could say that somebody is more or less adult "if he resembles an adult". Likewise a park is more or less large "if it resembles a large park". In general: v is more or less A if v resembles a u that is A. Hence v is more or less A if the intersection of A and Rv is not empty. Or to state it more fuzzily: y is more or less A to the degree to which Rv and A overlap, i.e.

$$\textsf{more or less } A(v) = R \uparrow_{\mathcal{T}} A(v) = \sup_{u \in U} \mathcal{T}(A(u), R(u, v))$$

Figure 1: Membership and non-membership functions of A and $R{\uparrow}A$

The L-fuzzy set $R \uparrow_{\mathcal{T}} A$ is called the direct image of the L-fuzzy set A under the L-fuzzy relation R, taken by means of a t-norm $\mathcal{T}$ on L.

For the representation of very an analogous scheme can be used. Indeed: if all men resembling Alberik in height are tall, then Alberik must be very tall. Likewise Krista is very kind "if everyone resembling Krista is kind". In general: v is very A if all u resembling v are A. Hence y is very A if Rv is included in A. To state it more fuzzily: v is very A to the degree to which Rv is included in A, i.e.

$$\textsf{very } A(v) = R \downarrow_{\mathcal{I}} A(v) = \inf_{u \in U} \mathcal{I}(R(u, v), A(u))$$

The L-fuzzy set $R \downarrow_{\mathcal{T}} A$ is called the superdirect image of the L-fuzzy set A under the L-fuzzy relation R, taken by means of an implicator $\mathcal{I}$ on L. Under the natural assumption that R is reflexive (every object is approximately equal to itself to the highest degree), semantical entailment holds. As mentioned in the introduction, since IFSs are also L-fuzzy sets, a representation for more or less and very is readily obtained. Figure 1 depicts the membership function μ_A and non-membership function ν_A of an IFS A. A is modified by taking the direct image by means of $\mathcal{T}_{\text{W}}$ under an IF relation R with a membership function based on the general shape $\mathbb{S}$-membership function

$$\mathbb{S}(x; \alpha, \gamma) = \begin{cases} 0, & x \leq \alpha \\ \frac{2(x-\alpha)^2}{(\gamma-\alpha)^2}, & \alpha \leq x \leq (\alpha+\gamma)/2 \\ 1 - \frac{2(x-\gamma)^2}{(\gamma-\alpha)^2}, & (\alpha+\gamma)/2 \leq x \leq \gamma \\ 1, & \gamma \leq x \end{cases}$$

for x, α and γ in $\mathbb{R}$ and $\alpha < \gamma$. Specifically R is defined as

$$\mu_R(u, v) = \begin{cases} \mathbb{S}(u; v-20, v-5) & \text{if } u \leq v-5 \\ 1 & \text{if } v-5 < u < v+5 \\ 1 - \mathbb{S}(u; v+5, v+20) & \text{if } v+5 \leq u \end{cases}$$

and $\nu_R(u, v) = 1 - \mu_R(u, v)$, for all u and v in U. This results in the membership and the non-membership function for the modified IFS $R{\uparrow}A$ depicted in Figure 1. As Figure 2 illustrates, the modification does not preserve the local hesitation: depending on its context, the hesitation degree of u in A increases, decreases or remains unaltered when passing to $R{\uparrow}A$. On the global level however, the overall hesitation seems to be invariant. In the next section we will illustrate that this does not hold in general.

Figure 2: Hesitation

4. DIVIDE AND CONQUER

As far as the authors are aware, the only other existing approach to the modification of linguistic terms modeled by IFSs is due to De, Biswas and Roy [3]. They proposed an extension of Zadeh's representation; it is based on the so–called product of IFSs A and B defined as $A \cdot B(u) = \mathcal{T}_{\mathrm{P}}(A(u), B(u))$, for all u in U. One can easily verify that

$$A^2(u) = (\mu_A(u)^2, 1 - (1 - \nu_A(u))^2) \tag{1}$$

in which A^2 is used as a shorthand notation for $A \cdot A$. Furthermore for $A^{\frac{1}{2}}$ defined in a similar manner as

$$A^{\frac{1}{2}}(u) = (\mu_A(u)^{\frac{1}{2}}, 1 - (1 - \nu_A(u))^{\frac{1}{2}}) \tag{2}$$

one can verify that $A^{\frac{1}{2}} \cdot A^{\frac{1}{2}} = A$ which justifies the notation. Entirely in the line of Zadeh's work, in [3] the authors propose to use $A^{\frac{1}{2}}$ and A^2 for the representation of more or less and very respectively. As a consequence, the drawbacks listed in Section 3 are also inherited, making the approach less interesting from the semantical point of view.

Nevertheless Equations (1) and (2) reveal some interesting semantical clues. Indeed, these formulas actually suggest to model very A by (very μ_A, not (very not ν_A)) and more or less A by (more or less μ_A, not (more or less not ν_A)). As such it is an example of what we have called the divide–and–conquer approach. The resulting expressions for the non–membership functions are clearly more complicated than those for the membership functions; they stem from the observation that the complement of the non–membership function can be interpreted loosely as a kind of second membership function.

With these considerations in mind, it is worthwhile to examine how divide–and–conquer behaves for the approach inspired by L–fuzzy relations. The following propositions show that, under certain conditions on the connectives used in the formulas, we can establish a meaningful representation for the modification of the whole in terms of that of its constituting parts. The proof of these propositions relies on the following lemma:

Lemma 1 *Let $(a_i, b_i)_{i \in I}$ be a family in L^*.*

$$\sup_{i \in I}(a_i, b_i) = \left(\sup_{i \in I} a_i, \inf_{i \in I} b_i\right)$$

$$\inf_{i \in I}(a_i, b_i) = \left(\inf_{i \in I} a_i, \sup_{i \in I} b_i\right)$$

Proposition 1 *Let $\mathcal{T}$ be a t–representable t–norm on L^* such that $\mathcal{T} = (T, S)$, N an involutive negator on $[0,1]$, and I the S–implicator on $[0,1]$ defined by $I(x, y) = S(N(x), y)$. Furthermore let $A \in \mathcal{IFS}(U)$, $R \in \mathcal{IFS}(U \times U)$. Then:*

$$(\mu_R{\uparrow}_T \mu_A, (co_N \nu_R){\downarrow}_I \nu_A) = R{\uparrow}_{\mathcal{T}} A$$

Under the natural assumption that R is reflexive, we have $co_N(\nu_R){\downarrow}_I \nu_A \subseteq \nu_A$ and $\mu_A \subseteq \mu_R{\uparrow}_T \mu_A$. If ν_A is the constant $[0,1] - \{0\}$ mapping, modification of the non-membership function will have no effect. Any change in the membership function will therefore give rise to a decrease of the overall hesitation. Note that this seems natural: the hesitation to call objects A might be greater than the hesitation to call them more or less A.

For the dual proposition we need the fuzzy standard complement, defined as $co\mu_A(u) = 1 - \mu_A(u)$, for every fuzzy set μ_A in U and for all u in U.

Proposition 2 *Let $\mathcal{S}$ be a t-representable t–conorm on L^* such that $\mathcal{S} = (S, T)$, $\mathcal{N}$ an involutive negator on L^*, and $\mathcal{I}$ the S–implicator on L^* defined by $\mathcal{I}(x, y) = \mathcal{S}(\mathcal{N}(x), y)$. Let N be the fuzzy negator induced by $\mathcal{N}$. Furthermore let $A \in \mathcal{IFS}(U)$, $R \in \mathcal{IFS}(U \times U)$. Then:*

$$((co\nu_R){\downarrow}_I \mu_A, co(co_N \mu_R){\uparrow}_T \nu_A) = R{\downarrow}_{\mathcal{I}} A$$

A first remarkable observation is that in both propositions on the "fuzzy level" the images are taken under the membership function μ_R, or something semantically very much related such as the N-complement of the non-membership function ν_R or once even the standard complement of the N-complement of μ_R.

As Proposition 1 indicates, taking the IF direct image ("more or less") involves both a fuzzy direct image ("more or less") and a fuzzy superdirect image ("very"). A dual observation can be made for Proposition 2. Interestingly enough, De, Biswas and Roy [3] do not use both hedges at the same time, but their approach involves negation of the non-membership function. Possible connections between intensifying hedges (like very) and weakening hedges (such as more or less) by means of negation have already intrigued several researchers. In [2] the meaning of "not overly bright" is described as "rather underly bright" which gives rise (albeit simplified) to the demand for equality of the mathematical representations for not very P and more or less not P. Under certain conditions on the connectives involved, it can be verified that the model described in Section 3 indeed behaves in this way. Specifically it holds that

$$co_N\nu_R{\downarrow_I}\nu_A = co_N(co_N\nu_R{\uparrow_T}(co_N\nu_A))$$
$$co(co_N\mu_R){\uparrow_T}\nu_A = co_N(co(co_N\mu_R{\downarrow_I}co_N\nu_A)$$

which reveals the semantical link between both approaches.

5. CONCLUSION

Partly due to existing studies on connectives in the lattice L^*, the context based relational approach for the representation of linguistic hedges is readily obtained in an intuitionistic fuzzy setting. The resulting modification is not hesitation invariant on the local nor the global level. Under certain conditions on the connectives used in the formulas of direct and superdirect image, a meaningful representation for the modification of the whole in terms of that of its constituting parts is established. This is not only interesting from the computational point of view, but also helps us to gain more insight in the semantics of the linguistic modification process.

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