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ELSEVIER International Journal of Approximate Reasoning 35 (2004) 55–95

INTERNATIONAL JOURNAL OF APPROXIMATE REASONING

www.elsevier.com/locate/ijar

Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification, application

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Received 1 August 2002; accepted 1 July 2003

Abstract

With the demand for knowledge-handling systems capable of dealing with and distinguishing between various facets of imprecision ever increasing, a clear and formal characterization of the mathematical models implementing such services is quintessential. In this paper, this task is undertaken simultaneously for the definition of implication within two settings: first, within intuitionistic fuzzy set theory and secondly, within interval-valued fuzzy set theory. By tracing these models back to the underlying lattice that they are defined on, on one hand we keep up with an important tradition of using algebraic structures for developing logical calculi (e.g. residuated lattices and MV algebras), and on the other hand we are able to expose in a clear manner the two models' formal equivalence. This equivalence, all too often neglected in literature, we exploit to construct operators extending the notions of classical and fuzzy implication on these structures; to initiate a meaningful classification framework for the resulting operators, based on logical and extra-logical criteria imposed on them; and finally, to re(de)fine the intuititive ideas giving rise to both approaches as models of imprecision and apply them in a practical context.

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URL: http://fuzzy.UGent.be.

0888-613X/$ - see front matter
doi:10.1016/S0888-613X(03)00072-0

Keywords: Intuitionistic fuzzy set theory; Interval-valued fuzzy set theory; Indeterminacy; Implicators; Smets–Magrez axioms; Residuated lattices; MV-algebras; Knowledge-based systems

1. Introduction

Intuitionistic fuzzy sets [1] and interval-valued fuzzy sets ([54,67] and more recently, [58]) are two intuitively straightforward extensions of Zadeh's fuzzy sets [66], that were conceived independently to alleviate some of the drawbacks of the latter. Henceforth, for notational ease, we abbreviate "intuitionistic fuzzy set" to IFS and "interval-valued fuzzy set" to IVFS. IFS theory basically defies the claim that from the fact that an element x "belongs" to a given degree (say μ) to a fuzzy set A, naturally follows that x should "not belong" to A to the extent $1 - \mu$, an assertion implicit in the concept of a fuzzy set. On the contrary, IFSs assign to each element of the universe both a degree of membership μ and one of non-membership v such that $\mu + v \leqslant 1$, thus relaxing the enforced duality $v = 1 - \mu$ from fuzzy set theory. Obviously, when $\mu + v = 1$ for all elements of the universe, the traditional fuzzy set concept is recovered. IFSs owe their name [4] to the fact that this latter identity is weakened into an inequality, in other words: a denial of the law of the excluded middle occurs, one of the main ideas of intuitionism. [1]

IVFS theory emerged from the observation that in a lot of cases, no objective procedure is available to select the crisp membership degrees of elements in a fuzzy set. It was suggested to alleviate that problem by allowing to specify only an interval $[\mu_1, \mu_2]$ to which the actual membership degree is assumed to belong. A related approach, second-order fuzzy set theory, also introduced by Zadeh [67], goes one step further by allowing the membership degrees themselves to be fuzzy sets in the unit interval; this extension is not considered in this paper.

Both approaches, IFS and IVFS theory, have the virtue of complementing fuzzy sets, that are able to model *vagueness*, with an ability to model *uncertainty* as well. [2] IVFSs reflect this uncertainty by the length of the interval membership degree $[\mu_1, \mu_2]$, while in IFS theory for every membership degree

[1] The term "intuitionistic" is to be read in a "broad" sense here, alluding loosely to the denial of the law of the excluded middle on element level (since $\mu + v < 1$ is possible). A "narrow", graded extension of intuitionistic logic proper has also been proposed and is due to Takeuti and Titani [57]—it bears no relationship to Atanassov's notion of IFS theory.

[2] In these pages, we juxtapose "vagueness" and "uncertainty" as two important aspects of imprecision. Some authors [45,47,60] prefer to speak of "non-specificity" and reserve the term "uncertainty" for the global notion of imprecision.

(μ, ν), the value $\pi = 1 - \mu - \nu$ denotes a measure of non-determinacy (or undecidedness).

Each approach has given rise to an extensive literature covering their respective applications, but surprisingly very few people seem to be aware of their equivalence, stated first in [2] and later in [31,63]. Indeed, take any IVFS A in a universe X, and assume that the membership degree of x in A is given as the interval $[\mu_1, \mu_2]$. Obviously, $\mu_1 + 1 - \mu_2 \leqslant 1$, so by defining $\mu = \mu_1$ and $\nu = 1 - \mu_2$ we obtain a valid membership and non-membership degree for x in an IFS A'. Conversely, starting from any IFS A' we may associate to it an IVFS A by putting, for each element x, the membership degree of x in A equal to the interval $[\mu, 1 - \nu]$ with again (μ, ν) the pair of membership/non-membership degrees of x in A'. As a consequence, a considerable body of work has been duplicated by adepts of either theory, or worse, is known to one group and ignored by the other. Therefore, regardless of the meaning (semantics) that one likes his or her preferred approach to convey, it is worthwhile to develop the underlying theory in a framework as abstract and general as possible. Lattices seem to lend themselves extremely well to that purpose; indeed it is common practice to interpret them as evaluation sets from which truth values are drawn and to use them as a starting point for developing logical calculi. Let us apply this strategy to the formal treatment of IVFSs and IFSs: we will describe them as special instances of Goguen's L-fuzzy sets, [3] where the appropriate evaluation set will be the bounded lattice $(L^*, \leqslant_{L^*})$ defined as [14]:

Definition 1 (*Lattice* $(L^*, \leqslant_{L^*})$)

$$L^* = \{(x_1, x_2) \in [0,1]^2 | x_1 + x_2 \leqslant 1\}$$
$$(x_1, x_2) \leqslant_{L^*} (y_1, y_2) \iff x_1 \leqslant y_1 \text{ and } x_2 \geqslant y_2$$

The units of this lattice are denoted $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$. A special subset of L^*, called the diagonal D, is defined by $D = \{(x_1, x_2) \in [0,1]^2 | x_1 + x_2 = 1\}$. The shaded area in Fig. 1 is the set of elements $x = (x_1, x_2)$ belonging to L^*.

Note. This definition favours IFSs as they are readily seen to be L-fuzzy sets w.r.t. this lattice, while for IVFSs a transformation from $(x_1, x_2) \in L^*$ to the interval $[x_1, 1 - x_2]$ must be performed beforehand; this decision reflects the background of the authors. Nevertheless, it is important to realize that nothing stands in our way to define equivalently:

$$L^I = \{(x_1, x_2) \in [0,1]^2 | x_1 \leqslant x_2\}$$

[3] Let $(L, \leqslant_L)$ be a complete lattice. An L-fuzzy set in U is an $U \to L$ mapping [36].

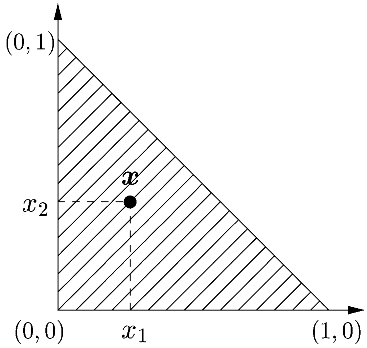

Fig. 1. Graphical representation of the set L^*.

$$(x_1, x_2) \leqslant_{L^I} (y_1, y_2) \quad \iff \quad x_1 \leqslant y_1 \text{ and } x_2 \leqslant y_2$$

and develop the theory in terms of $(L^I, \leqslant_{L^I})$. For compliance with the existing literature, we denote the class of L^*-fuzzy sets in a universe U by $\mathscr{F}_{L^*}(U)$.

Note. In this paper, if $x \in L^*$, we refer to its and first and second components by x_1 and x_2 respectively. In case we want to refer to the individual components of an expression like $f(x)$, where in this case for instance f is an $L^* \to L^*$ mapping, we write $\text{pr}_1 f(x)$ and $\text{pr}_2 f(x)$, where the projections pr_1 and pr_2 map an ordered pair (in this case an element of L^*) to its first and second component, respectively.

The lattice $(L^*, \leqslant_{L^*})$ is a complete lattice: for each $A \subseteq L^*$, $\sup A = (\sup\{x \in [0, 1] | (\exists y \in [0, 1])((x, y) \in A)\}, \inf\{y \in [0, 1] | (\exists x \in [0, 1])((x, y) \in A)\})$ and $\inf A = (\inf\{x \in [0, 1] | (\exists y \in [0, 1])((x, y) \in A)\}, \sup\{y \in [0, 1] | (\exists x \in [0, 1])((x, y) \in A)\})$.

As is well known, every lattice $(L, \leqslant)$ has an equivalent definition as an algebraic structure $(L, \wedge, \vee)$ where the meet operator $\wedge$ and the join operator $\vee$ are linked to the ordering $\leqslant$ by the following equivalence, for $a, b \in L$:

$$a \leqslant b \iff a \vee b = b \iff a \wedge b = a$$

The operators $\wedge$ and $\vee$ on $(L^*, \leqslant_{L^*})$ are defined as follows, for $(x_1, y_1), (x_2, y_2) \in L^*$:

$$(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2))$$
$$(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2))$$

This algebraic structure will be the basis for our subsequent investigations. In the next section, entitled "Preliminaries" the most important operations on $(L^*, \leqslant_{L^*})$ are defined, notably: triangular norms and conorms, negators and implicators. They model the basic logical operations of conjunction, disjunction, negation and implication. Implicators on L^* will be the main point of interest in the remainder of the paper: in Section 3 we review construction techniques for them, Section 4 examines their classification w.r.t. a number of criteria imposed on them while Section 5 embeds the results into the frameworks of well-known logical calculi such as residuated lattices and MV algebras. Section 6 then puts the focus back on the models that we started out with: IFSs and IVFSs, and describes their applicability in the field of approximate reasoning. Future opportunities and challenges are also discussed in that section.

2. Preliminaries

In the literature on IFSs and IVFSs, several methods for constructing connectives have emerged, their rationale typically based on specific considerations taken in the light of the actual framework for which they were developed. While most of them have the advantage of being readily understood by anyone familiar with that framework, they are not always the most general nor the most suitable ones that could be defined. Therefore, to put matters in as wide as possible a perspective, in this and the next section, we introduce logical connectives simply as algebraic mappings on L^*, regardless of their interpretation in the context of a specific model. We recall the definitions of the main logical operations in $(L^*, \leqslant_{L^*})$, as well as some of the representation results established earlier and obtained in the framework of an extensive study on intuitionistic fuzzy triangular norms and conorms [27–29].

Definition 2 (*Negator on* L^*). A negator on L^* is any decreasing $L^* \to L^*$ mapping $\mathscr{N}$ satisfying $\mathscr{N}(0_{L^*}) = 1_{L^*}$, $\mathscr{N}(1_{L^*}) = 0_{L^*}$. If $\mathscr{N}(\mathscr{N}(x)) = x\ \forall x \in L^*$, $\mathscr{N}$ is called an involutive negator.

The mapping $\mathscr{N}_s$, defined as $\mathscr{N}_s(x_1,x_2) = (x_2,x_1)\ \forall (x_1,x_2) \in L^*$, will be called the *standard negator*. Involutive negators on L^* can always be related to an involutive negator on $[0,1]$, as the following theorem shows [29].

Theorem 1. *Let* $\mathscr{N}$ *be an involutive negator on* L^*, *and let the* $[0,1] \to [0,1]$ *mapping* N *be defined by, for* $a \in [0,1]$, $N(a) = \mathrm{pr}_1 \mathscr{N}(a, 1-a)$. *Then for all* $(x_1,x_2) \in L^* : \mathscr{N}(x_1,x_2) = (N(1-x_2), 1-N(x_1))$.

Since $\leqslant_{L^*}$ is a partial ordering, an order-theoretic definition of conjunction and disjunction on L^* as triangular norms and conorms, *t*-norms and *t*-conorms for short, respectively, arises quite naturally:

Definition 3 (*Triangular Norm on* L^*). A *t*-norm on L^* is any increasing, commutative, associative $(L^*)^2 \to L^*$ mapping $\mathscr{T}$ satisfying $\mathscr{T}(1_{L^*}, x) = x$ for all $x \in L^*$.

Definition 4 (*Triangular Conorm on L^**). A t-conorm on L^* is any increasing, commutative, associative $(L^*)^2 \to L^*$ mapping $\mathcal{S}$ satisfying $\mathcal{S}(0_{L^*}, x) = x$, for all $x \in L^*$.

Obviously, the greatest t-norm with respect to the ordering $\leqslant_{L^*}$ is Min, defined by $\mathrm{Min}\,(x, y) = x \wedge y$; the smallest t-conorm w.r.t. $\leqslant_{L^*}$ is Max, defined by $\mathrm{Max}\,(x, y) = x \vee y$ for all $x, y \in L^*$. Note that it does not hold that for all $x, y \in L^*$, either $\mathrm{Min}\,(x, y) = x$ or $\mathrm{Min}\,(x, y) = y$. For instance, $\mathrm{Min}\,((0.1, 0.3), (0.2, 0.4)) = (0.1, 0.4)$.

Involutive negators on L^* are always linked to an associated fuzzy connective (a negator on $[0, 1]$); the same does not always hold true for t-norms and t-conorms, however. We therefore have to introduce the following definition [16]:

Definition 5 (*t-representability*). A t-norm $\mathcal{T}$ on L^* (respectively t-conorm $\mathcal{S}$) is called t-representable if there exists a t-norm T and a t-conorm S on $[0, 1]$ (respectively a t-conorm S' and a t-norm T' on $[0, 1]$) such that, for $x = (x_1, x_2)$, $y = (y_1, y_2) \in L^*$,

$$\mathcal{T}(x, y) = (T(x_1, y_1), S(x_2, y_2))$$
$$\mathcal{S}(x, y) = (S'(x_1, y_1), T'(x_2, y_2))$$

T and S (respectively S' and T') are called the representants of $\mathcal{T}$ (respectively $\mathcal{S}$).

Example 1. Consider the following mappings on L^*:

$$\mathcal{S}_1(x, y) = (x_1 + y_1 - x_1 y_1, x_2 y_2)$$
$$\mathcal{S}_2(x, y) = \begin{cases} x & \text{if } y = 0_{L^*} \\ y & \text{if } x = 0_{L^*} \\ (\max(1 - x_2, 1 - y_2), \min(x_2, y_2)) & \text{else} \end{cases}$$

It is easily verified that they are t-conorms. The first one is t-representable with the probabilistic sum and algebraic product on $[0, 1]$ as representants. It is an extension of the probabilistic sum t-conorm to L^*. The second one is not t-representable, since its first component depends also on x_2 and y_2. It is an extension of the max t-conorm to L^*.

The theorem below states the conditions under which a pair of connectives on $[0, 1]$ gives rise to a t-representable t-norm (t-conorm) on L^*.

Theorem 2 [16]. *Given a t-norm T and t-conorm S on $[0, 1]$ satisfying $T(a, b) \leqslant 1 - S(1 - a, 1 - b)$ for all $a, b \in [0, 1]$, the mappings $\mathcal{T}$ and $\mathcal{S}$ defined by, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in L^*:*

$$\mathscr{T}(x,y) = (T(x_1,y_1), S(x_2,y_2)),$$
$$\mathscr{S}(x,y) = (S(x_1,y_1), T(x_2,y_2)),$$

are a t-norm and a t-conorm on L^*, respectively.

Note. The discovery of a mapping like S_2, first mentioned in [16], rules out the conjecture, implicit in most of the existing literature (see e.g. [7,14,35,41]), that interval-valued or intuitionistic fuzzy t-norms and t-conorms are necessarily characterized by a pair of fuzzy connectives on which some condition (cf. Theorem 2) is imposed to assure that the result of an operation belongs to the evaluation set. Moreover, as we shall see in Section 4, implicators based on t-representable operators do not inherit as much of the desirable properties of their fuzzy counterparts as we would like them to, a defect that can be mended by considering non-t-representable extensions for the implicator construction.

The dual of a t-norm $\mathscr{T}$ on L^* (t-conorm $\mathscr{S}$) w.r.t. a negator $\mathscr{N}$ is the mapping $\mathscr{T}^*$ (respectively $\mathscr{S}^*$) defined by, for $x, y \in L^*$,

$$\mathscr{T}^*(x,y) = \mathscr{N}(\mathscr{T}(\mathscr{N}(x), \mathscr{N}(y)))$$
$$(\text{respectively } \mathscr{S}^*(x,y) = \mathscr{N}(\mathscr{S}(\mathscr{N}(x), \mathscr{N}(y))))$$

It can be verified that $\mathscr{T}^*$ is a t-conorm and $\mathscr{S}^*$ is a t-norm on L^*. Moreover, the dual t-norm (t-conorm) with respect to an involutive negator $\mathscr{N}$ on L^* of a t-representable t-conorm (t-norm) is t-representable [29].

In [29] a representation theorem for t-norms on L^* meeting a number of criteria was formulated and proven.

Theorem 3. *$\mathscr{T}$ is a continuous t-norm on L^* satisfying*

- $(\forall x \in L^* \setminus \{0_{L^*}, 1_{L^*}\})(\mathscr{T}(x,x) <_{L^*} x)$ (*archimedean property*)
- $(\exists x, y \in L^*)$ $(x_1 \neq 0$ *and* $x_2 \neq 0$ *and* $y_1 \neq 0$ *and* $y_2 \neq 0$ *and* $\mathscr{T}(x,y) = 0_{L^*})$ (*strong nilpotency*)
- $(\forall x, y, z \in L^*)(\mathscr{T}(x,z) \leqslant_{L^*} y \Longleftrightarrow z \leqslant_{L^*} \sup\{\gamma \in L^* | \mathscr{T}(x,\gamma) \leqslant_{L^*} y\})$ (*residuation principle*)
- $(\forall x, y \in D)(\sup\{\gamma \in L^* | \mathscr{T}(x,\gamma) \leqslant_{L^*} y\} \in D)$
- $\mathscr{T}((0,0),(0,0)) = 0_{L^*}$

  if and only if there exists an increasing continuous permutation φ of $[0,1]$ such that for all $x, y \in L^*$,

$$\mathscr{T}(x,y) = (\varphi^{-1}(\max(0, \varphi(x_1) + \varphi(y_1) - 1)), 1 - \varphi^{-1}(\max(0, \varphi(x_1) + \varphi(1 - y_2) - 1, \varphi(y_1) + \varphi(1 - x_2) - 1)))$$

or equivalently, there exists a continuous increasing permutation Φ of L^* such that Φ^{-1} is increasing and such that $\mathscr{T} = \Phi^{-1} \circ \mathscr{T}_W \circ (\Phi \circ \mathrm{pr}_1, \Phi \circ \mathrm{pr}_2)$, where $\mathscr{T}_W$, the Łukasiewicz t-norm on L^*, is defined by, for $x, y \in L^*$:

$$\mathscr{T}_W(x,y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + 1 - y_1, y_2 + 1 - x_1))$$

The list of imposed conditions on $\mathscr{T}$ seems overwhelming, but on closer inspection the relationship with the analogous result in fuzzy set theory (representation of continuous, archimedean, nilpotent t-norms on $[0,1]$, see e.g. [46]) becomes obvious, so it is justified to state that Theorem 3 constitutes a conservative extension of that result. A generalization of Theorem 3 can be found in [30].

The final and for our present purposes most important construct is that of an implicator on L^*. Our definition includes a very wide class of mappings on L^*; the task of classification (Section 3) will be to select from this class those implicators that are, in a way, the most appropriate ones.

Definition 6 (*Implicator on* L^*). An implicator on L^* is any $(L^*)^2 \to L^*$-mapping $\mathscr{I}$ satisfying $\mathscr{I}(0_{L^*}, 0_{L^*}) = 1_{L^*}$, $\mathscr{I}(1_{L^*}, 0_{L^*}) = 0_{L^*}$, $\mathscr{I}(0_{L^*}, 1_{L^*}) = 1_{L^*}$, $\mathscr{I}(1_{L^*}, 1_{L^*}) = 1_{L^*}$. Moreover we require $\mathscr{I}$ to be decreasing in its first, and increasing in its second component.

3. Construction of Implicators on L^*

The purpose of this section is to give the reader some feeling for the sheer multitude of connectives that fit Definition 6, by providing a number of illustrative examples, and also to arm ourselves sufficiently well for the classification task that awaits us in the next section, by putting some structure into the class of implicators on L^*: as a point of departure, we extend the common notions of S- and R-implicators from fuzzy set theory to L^* [16,27], an approach that has the virtue of being the algebraically most straightforward and flexible one (w.r.t. classification). The story does not end there, however, as the eclectic literature on intuitionistic fuzzy and interval-valued connectives has bequeathed us with some operators outside the above-mentioned classes but in line with Definition 6 and with varying usefulness.

3.1. Strong implicators on L^*

Strong implicators, or shortly S-implicators, on the unit interval emerged by paraphrasing the equivalence between the formulas $P \to Q$ and $\neg P \vee Q$ in binary propositional logic using a negator and a t-conorm. A straightforward extension to L^* presents itself as follows:

Definition 7 (*S-implicator on* L^*). Let $\mathscr{S}$ be a t-conorm and $\mathscr{N}$ a negator on L^*. The S-implicator generated by $\mathscr{S}$ and $\mathscr{N}$ is the mapping $\mathscr{I}_{\mathscr{S},\mathscr{N}}$ defined as, for $x, y \in L^*$:

$$\mathscr{I}_{\mathscr{S},\mathscr{N}}(x,y) = \mathscr{S}(\mathscr{N}(x),y)$$

If $\mathscr{S}$ is t-representable, $\mathscr{I}_{\mathscr{S},\mathscr{N}}$ is called a t-representable S-implicator on L^*.

It can be verified that the resulting construct satisfies Definition 6 regardless of $\mathscr{N}$ and $\mathscr{S}$. Below we present some common examples of S-implicators on L^*; for every operator thus defined we also quote the corresponding connective on the unit interval that this implicator extends. Note especially how a single implicator on $[0,1]$ gives way to several extensions, with—as will become clear in the next section—significantly differing properties.

Example 2. Let $\mathscr{S} = \text{Max}$ and $\mathscr{N} = \mathscr{N}_s$. Then

$$\mathscr{I}_{\text{Max},\mathscr{N}_s}(x,y) = (\max(x_2,y_1),\min(x_1,y_2))$$

$\mathscr{I}_{\text{Max},\mathscr{N}_s}$ is an extension of the Kleene–Dienes implicator on $[0,1]$, $I_b(x,y) = \max(1-x,y)$ for all $x,y \in [0,1]$. Since Max is the smallest t-conorm on L^*, $\mathscr{I}_{\text{Max},\mathscr{N}_s}(x,y) \leqslant_{L^*} \mathscr{I}_{\mathscr{S},\mathscr{N}_s}(x,y)$ for arbitrary t-conorm $\mathscr{S}$ on L^* and for all $x,y \in L^*$. This implicator occurred in literature previously in the work of e.g. Atanassov and Gargov [3,5] on IFSs.

Example 3. Let $\mathscr{S}(x,y) = (\min(1,x_1+y_1),\max(0,x_2+y_2-1))$ for all $x,y \in L^*$ and $\mathscr{N} = \mathscr{N}_s$. Then

$$\mathscr{I}_{\mathscr{S},\mathscr{N}_s}(x,y) = (\min(1,x_2+y_1),\max(0,x_1+y_2-1))$$

$\mathscr{I}_{\mathscr{S},\mathscr{N}_s}$ is an extension of the Łukasiewicz implicator on $[0,1]$, $I_a(x,y) = \min(1,1-x+y)$ for all $x,y \in [0,1]$.

Example 4. Let $\mathscr{S}(x,y) = (\min(1,x_1+1-y_2,y_1+1-x_2),\max(0,x_2+y_2-1))$ for all $x,y \in L^*$ (i.e. $\mathscr{S} = \mathscr{S}_W$, the dual of the Łukasiewicz t-norm $\mathscr{T}_W$) and $\mathscr{N} = \mathscr{N}_s$. Then

$$\mathscr{I}_{\mathscr{S}_W,\mathscr{N}_s}(x,y) = (\min(1,y_1+1-x_1,x_2+1-y_2),\max(0,x_1+y_2-1))$$

$\mathscr{I}_{\mathscr{S}_W,\mathscr{N}_s}$ is another extension of the Łukasiewicz implicator on $[0,1]$. It is also an example of a non-t-representable S-implicator.

3.2. Residual implicators on L^*

Another way of defining implication in classical logic is to consider the equivalence

$$P \to Q \equiv \sup\{X \in \{0,1\} | P \wedge X \leqslant Q\}$$

and fuzzify it, using a t-norm, to obtain the definition of residual implicators on $[0,1]$, or R-implicators for short.

Definition 8 (*R-implicator on L^**). Let $\mathscr{T}$ be a *t*-norm on L^*. The R-implicator generated by $\mathscr{T}$ is the mapping $\mathscr{I}_{\mathscr{T}}$ defined as, for $x, y \in L^*$:

$$\mathscr{I}_{\mathscr{T}}(x, y) = \sup\{\gamma \in L^* | \mathscr{T}(x, \gamma) \leqslant_{L^*} y\}$$

If $\mathscr{T}$ is *t*-representable, $\mathscr{I}_{\mathscr{T}}$ is called a *t*-representable R-implicator on L^*.

Again, the above-defined mappings are implicators on L^* in the sense of Definition 6. Some of them have occurred previously in literature; for instance, in [41], Jenei already introduced the class of *t*-representable R-implicators on L^* in the specific setting of IVFSs.

Due to the supremum operation appearing in their definition, it is not always straightforward to derive an explicit form for R-implicators on L^*, as the examples below show.

Example 5. Let $\mathscr{T}(x,y) = \text{Min}(x,y) = (\min(x_1,y_1), \max(x_2,y_2))$ for all $x,y \in L^*$. Then

$$\mathscr{I}_{\text{Min}}(x, y) = \sup\{\gamma \in L^* | (\min(x_1, \gamma_1), \max(x_2, \gamma_2)) \leqslant_{L^*} y\}$$

We now derive an explicit formula for $\mathscr{I}_{\text{Min}}$:

- If $x_1 \leqslant y_1$ and $x_2 \geqslant y_2$, then $\min(x_1, \gamma_1) \leqslant x_1 \leqslant y_1$ $\forall \gamma_1 \in [0,1]$ and $\max(x_2, \gamma_2) \geqslant x_2 \geqslant y_2$ $\forall \gamma_2 \in [0,1]$. Hence, in that case, $\mathscr{I}_{\text{Min}}(x, y) = 1_{L^*}$.
- If $x_1 \leqslant y_1$ and $x_2 < y_2$, then still $\min(x_1, \gamma_1) \leqslant x_1 \leqslant y_1$ $\forall \gamma_1 \in [0,1]$, but $\max(x_2, \gamma_2) \geqslant y_2$ if and only if $\gamma_2 \geqslant y_2$, hence $\inf\{\gamma_2 \in [0,1] | \max(x_2, \gamma_2) \geqslant y_2\} = y_2$. We conclude that $\mathscr{I}_{\text{Min}}(x, y) = (1 - y_2, y_2)$.
- If $x_1 > y_1$ and $x_2 \geqslant y_2$, then still $\max(x_2, \gamma_2) \geqslant x_2 \geqslant y_2$ $\forall \gamma_2 \in [0,1]$, but $\min(x_1, \gamma_1) \leqslant y_1$ if and only if $\gamma_1 \leqslant y_1$, hence $\sup\{\gamma_1 \in [0,1] | \min(x_1, \gamma_1) \leqslant y_1\} = y_1$. We conclude that $\mathscr{I}_{\text{Min}}(x, y) = (y_1, 0)$.
- If $x_1 > y_1$ and $x_2 < y_2$, then

$$\sup\{\gamma_1 \in [0,1] | \min(x_1, \gamma_1) \leqslant y_1\} = y_1$$
$$\inf\{\gamma_2 \in [0,1] | \max(x_2, \gamma_2) \geqslant y_2\} = y_2$$

Since $y \in L^*$, we may conclude that $\mathscr{I}_{\text{Min}}(x, y) = (y_1, y_2)$.

To summarize, we obtain:

$$\mathscr{I}_{\text{Min}}(x, y) = \begin{cases} 1_{L^*} & \text{if } x_1 \leqslant y_1 \text{ and } x_2 \geqslant y_2 \\ (1 - y_2, y_2) & \text{if } x_1 \leqslant y_1 \text{ and } x_2 < y_2 \\ (y_1, 0) & \text{if } x_1 > y_1 \text{ and } x_2 \geqslant y_2 \\ (y_1, y_2) & \text{if } x_1 > y_1 \text{ and } x_2 < y_2 \end{cases}$$

$\mathscr{I}_{\text{Min}}$ is an extension of the Gödel implicator on $[0,1]$, defined by, for $x, y \in [0,1]$:

$$I_{\text{g}}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y \\ y & \text{otherwise} \end{cases}$$

Since Min is the greatest *t*-norm on L^*, $\mathscr{I}_{\text{Min}}$ is the smallest R-implicator on L^*.

Example 6. Let $\mathscr{T}(x,y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + y_2))$. Then

$$\mathscr{I}_{\mathscr{T}}(x,y) = \sup\{\gamma \in L^* | (\max(0, x_1 + \gamma_1 - 1), \min(1, x_2 + \gamma_2)) \leqslant_{L^*} y\}$$

To find an explicit formula for $\mathscr{I}_{\mathscr{T}}$, we distinguish between the following cases:

- If $x_1 \leqslant y_1$ and $x_2 \geqslant y_2$, then clearly $x_1 + \gamma_1 - 1 \leqslant y_1$ and $x_2 + \gamma_2 \geqslant y_2$ $\forall (\gamma_1, \gamma_2) \in L^*$. It follows easily that $\mathscr{I}_{\mathscr{T}}(x,y) = 1_{L^*}$.
- If $x_1 \leqslant y_1$ and $x_2 < y_2$, then still $x_1 + \gamma_1 - 1 \leqslant y_1$ $\forall \gamma_1 \in [0,1]$. The expression $x_2 + \gamma_2 \geqslant y_2$ is equivalent to $\gamma_2 \geqslant y_2 - x_2$. But $y_2 - x_2 > 0$. Hence $\inf\{\gamma_2 \in [0,1] | x_2 + \gamma_2 \geqslant y_2\} = y_2 - x_2$. So $\mathscr{I}_{\mathscr{T}}(x,y) = (1 - (y_2 - x_2), y_2 - x_2) = (1 - y_2 + x_2, y_2 - x_2)$.
- If $x_1 > y_1$ and $x_2 \geqslant y_2$ then $x_2 + \gamma_2 \geqslant y_2$ $\forall \gamma_2 \in [0,1]$. The condition $x_1 + \gamma_1 - 1 \leqslant y_1$ is equivalent to $\gamma_1 \leqslant 1 + y_1 - x_1$. But now $1 + y_1 - x_1 < 1$, so $\sup\{\gamma_1 \in [0,1] | x_1 + \gamma_1 - 1 \leqslant y_1\} = 1 + y_1 - x_1$. Hence $\mathscr{I}_{\mathscr{T}}(x,y) = (1 + y_1 - x_1, 0)$.
- If $x_1 > y_1$ and $x_2 < y_2$, then $x_1 + \gamma_1 - 1 \leqslant y_1$ is equivalent to $\gamma_1 \leqslant 1 + y_1 - x_1$, and $x_2 + \gamma_2 \geqslant y_2$ is equivalent to $\gamma_2 \geqslant y_2 - x_2$. Since we also require $\gamma_1 + \gamma_2 \leqslant 1$, we need to find the supremum (in L^*) of the set of (γ_1, γ_2)'s that satisfy the following array of inequalities:

$$\begin{cases} \gamma_1 \leqslant 1 + y_1 - x_1 \\ \gamma_2 \geqslant y_2 - x_2 \\ \gamma_1 + \gamma_2 \leqslant 1 \end{cases} \tag{1}$$

Fig. 2 shows the set of solutions (shaded area) to this array of inequalities graphically; depending on the position of x and y we have to distinguish between two possible situations, denoted (a) and (b) in the figure.

It is clear that in each case the supremum of the shaded area is equal to:

$$\mathscr{I}_{\mathscr{T}}(x,y) = (\min(1 - y_2 + x_2, 1 + y_1 - x_1), y_2 - x_2)$$

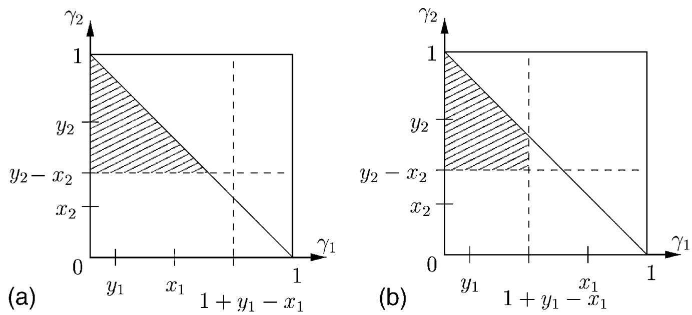

Fig. 2. (a) $1 - y_2 + x_2 < 1 + y_1 - x_1$; (b) $1 - y_2 + x_2 \geqslant 1 + y_1 - x_1$.

In summary we get:

$$\mathscr{I}_{\mathscr{T}}(x,y) = (\min(1, 1 + y_1 - x_1, 1 + x_2 - y_2), \max(0, y_2 - x_2))$$

$\mathscr{I}_{\mathscr{T}}$ is an extension of the Łukasiewicz implicator on $[0, 1]$ (see Examples 3 and 4).

Example 7. Let $\mathscr{T} = \mathscr{T}_W$, the Łukasiewicz t-norm on L^*. Then

$$\begin{aligned}&\mathscr{I}_{\mathscr{T}_W}(x,y)\\&\quad = \sup\{\gamma \in L^* | (\max(0, x_1 + \gamma_1 - 1), \min(1, x_2 + 1 - \gamma_1, \gamma_2 + 1 - x_1)) \leqslant_{L^*} y\}\end{aligned}$$

To find an explicit expression for $\mathscr{I}_{\mathscr{T}_W}$, let $x, y, \gamma \in L^*$. Then

$$\begin{aligned}&\mathscr{T}_W(x,\gamma) \leqslant_{L^*} y\\&\quad\Longleftrightarrow \max(0, x_1 + \gamma_1 - 1) \leqslant y_1 \quad\text{and}\quad \min(1, x_2 + 1 - \gamma_1, \gamma_2 + 1 - x_1) \geqslant y_2\\&\quad\Longleftrightarrow x_1 + \gamma_1 - 1 \leqslant y_1 \quad\text{and}\quad x_2 + 1 - \gamma_1 \geqslant y_2 \quad\text{and}\quad \gamma_2 + 1 - x_1 \geqslant y_2\\&\quad\Longleftrightarrow \gamma_1 \leqslant y_1 + 1 - x_1 \quad\text{and}\quad \gamma_1 \leqslant x_2 + 1 - y_2 \quad\text{and}\quad \gamma_2 \geqslant y_2 + x_1 - 1\\&\quad\Longleftrightarrow \gamma_1 \leqslant \min(1, y_1 + 1 - x_1, x_2 + 1 - y_2) \quad\text{and}\quad \gamma_2 \geqslant \max(0, y_2 + x_1 - 1)\end{aligned}$$

The last formula holds because γ is an element of L^*. Hence we obtain $\mathscr{I}_{\mathscr{T}_W}(x,y) = \sup\{\gamma \in L^* | \mathscr{T}_W(x,\gamma) \leqslant_{L^*} y\} = (\min(1, y_1 + 1 - x_1, x_2 + 1 - y_2), \max(0, y_2 + x_1 - 1))$.

Note especially that $\mathscr{I}_{\mathscr{T}_W} = \mathscr{I}_{\mathscr{S}_W, \mathscr{N}_s}$, and thus it also extends the Łukasiewicz implicator $\mathscr{I}_a$ on $[0, 1]$. This should not come as a surprise since in fuzzy logic the Łukasiewicz implicator is both an R- and an S-implicator.

3.3. Miscellaneous implicators and related operators on L^* outside the previous classes

The phrase "Implicators and Related Operators on L^*" in the title of this subsection owes to the fact that not all the "implicators" defined so far within the literature on IFSs and IVFSs meet the criteria set by Definition 6. It is definitely not our goal to produce an exhaustive list of all possible alternatives; we merely quote some of the more interesting examples.

Example 8 (*Two alternative extensions of Gödel implication*). In Example 5, we constructed an R-implicator on L^* that was an extension of I_g, the Gödel implicator (itself also an R-implicator) on $[0, 1]$. Below we outline two alternative generalizations of $\mathscr{I}_g$, neither of which is an R-implicator (or an S-implicator, for that matter) on L^*.

The first one was defined in [3] by Atanassov and Gargov as an implication operator for intuitionistic fuzzy logic; in the context of $(L^*, \leqslant_{L^*})$ it can be paraphrased as:

$$\mathscr{I}_{\mathrm{ag}}(x,y) = \begin{cases} 1_{L^*} & \text{if } x_1 \leqslant y_1 \\ (y_1, 0) & \text{if } x_1 > y_1 \quad \text{and} \quad x_2 \geqslant y_2 \\ (y_1, y_2) & \text{if } x_1 > y_1 \quad \text{and} \quad x_2 < y_2 \end{cases}$$

Let us start by proving that $\mathscr{I}_{\mathrm{ag}}$ is not an S-implicator; suppose $\mathscr{I}_{\mathrm{ag}}(x,y) = \mathscr{S}(\mathscr{N}(x),y)$ for all $x,y \in L^*$, $\mathscr{S}$ a t-conorm and $\mathscr{N}$ a negator on L^*. Since $\mathscr{S}(\mathscr{N}(x),0_{L^*}) = \mathscr{N}(x)$, we find

$$\mathscr{N}(x) = \mathscr{I}_{\mathrm{ag}}(x, 0_{L^*}) = \begin{cases} 1_{L^*} & \text{if } x_1 = 0 \\ 0_{L^*} & \text{otherwise} \end{cases}$$

Now put $x = (0.25, 0.45)$, then $\mathscr{N}(x) = 0_{L^*}$ and $\mathscr{S}(0_{L^*}, y) = y$ regardless of $\mathscr{S}$. But if e.g. $y = (0.1, 0.3)$, then $\mathscr{I}_{\mathrm{ag}}(x,y) = (0.1, 0) \neq y$. Thus, there does not exist such an $\mathscr{S}$ and hence $\mathscr{I}_{\mathrm{ag}}$ is not an S-implicator.

Secondly, suppose that $\mathscr{I}_{\mathrm{ag}}(x,y) = \sup\{\gamma \in L^* | \mathscr{T}(x,\gamma) \leqslant_{L^*} y\}$ for all $x,y \in L^*$, and $\mathscr{T}$ a t-norm on L^*. Let $x \in L^* \setminus D$ such that $x_1, x_2 > 0$, $\eta \in L^*$ such that $\eta <_{L^*} 1_{L^*}$ and $\eta_1 < x_1$ and $1 - x_1 > \eta_2 > 0$ (this is always possible since $x \notin D$). Then $\mathscr{T}(x,\eta) \leqslant_{L^*} (x_1, 1 - x_1)$ holds, [4] so $\mathrm{pr}_2\mathscr{T}(x,\eta) \geqslant 1 - x_1$. Similarly, $\mathscr{T}(\eta, x) \leqslant_{L^*} (\eta_1, 1 - \eta_1)$, so $\mathrm{pr}_2\mathscr{T}(x,\eta) = \mathrm{pr}_2\mathscr{T}(\eta,x) \geqslant 1 - \eta_1$. Thus,

$$\mathrm{pr}_2\mathscr{T}(x,\eta) \geqslant \max(1 - x_1, 1 - \eta_1)$$

Now put $y = (\eta_1, 1 - x_1)$, so $\mathrm{pr}_2\mathscr{T}(x,\eta) \geqslant 1 - x_1 = y_2$. On the other hand, $\mathrm{pr}_1\mathscr{T}(x,\eta) \leqslant \min(x_1, \eta_1) = \eta_1 = y_1$, and thus $\eta \in \{\gamma \in L^* | \mathscr{T}(x,\gamma) \leqslant_{L^*} y\}$. But then $\sup\{\gamma \in L^* | \mathscr{T}(x,\gamma) \leqslant_{L^*} y\} \geqslant_{L^*} \eta >_{L^*} (\eta_1, 1 - x_1) = y$, a contradiction since $\mathscr{I}_{\mathrm{ag}}(x,y) = y$. Hence $\mathscr{I}_{\mathrm{ag}}$ is not an R-implicator.

The second extension of $\mathscr{I}_{\mathrm{g}}$ we present here may be considered in some way its most genuine generalization to L^*. Defined by, for $x, y \in L^*$:

$$\mathscr{I}_{\mathrm{G}}(x,y) = \begin{cases} 1_{L^*} & \text{if } x \leqslant_{L^*} y \\ y & \text{otherwise} \end{cases}$$

it is however an implicator without a representation as an S- or R-implicator. To check this, suppose $\mathscr{I}_{\mathrm{G}}(x,y) = \mathscr{S}(\mathscr{N}(x),y)$ for all $x,y \in L^*$, $\mathscr{S}$ a t-conorm and $\mathscr{N}$ a negator on L^*. We find

$$\mathscr{N}(x) = \mathscr{I}_{\mathrm{G}}(x, 0_{L^*}) = \begin{cases} 1_{L^*} & \text{if } x = 0_{L^*} \\ 0_{L^*} & \text{otherwise} \end{cases}$$

Suppose now that $x \neq 0_{L^*}$ and $x \leqslant_{L^*} y <_{L^*} 1_{L^*}$. In that case we find $\mathscr{S}(\mathscr{N}(x),y) = \mathscr{S}(0_{L^*},y) = y <_{L^*} 1_{L^*} = \mathscr{I}_{\mathrm{G}}(x,y)$, a contradiction. Hence, $\mathscr{I}_{\mathrm{G}}$ is not an S-implicator.

[4] Indeed, since $x_1 \leqslant x_1$, $\mathscr{I}_{\mathrm{ag}}(x,(x_1, 1 - x_1)) = 1_{L^*} = \sup\{\gamma \in L^* | \mathscr{T}(x,\gamma) \leqslant_{L^*} (x_1, 1 - x_1)\}$, so for all $\gamma \in L^*$ such that $\gamma_2 > 0$, we have $\mathscr{T}(x,\gamma) \leqslant_{L^*} (x_1, 1 - x_1)$.

Suppose on the other hand that $\mathscr{I}_g(x,y) = \sup\{\gamma \in L^* | \mathscr{T}(x,\gamma) \leqslant_{L^*} y\}$ for all $x, y \in L^*$, and $\mathscr{T}$ a t-norm on L^*. Choose $x, y \in L^* \setminus D$ such that $x_1 > y_1$ and $x_2 \geqslant y_2 > 0$. Then $\mathscr{I}_g(x,y) = (y_1, y_2)$ by assumption. On the other hand, $\mathscr{I}_{\mathscr{T}}(x,y) \geqslant_{L^*} \mathscr{I}_{\mathrm{Min}}(x,y) = (y_1, 0)$ (see Example 5), since Min is the greatest t-norm on L^*. But $(y_1, 0) >_{L^*} (y_1, y_2)$, again a contradiction. Hence, $\mathscr{I}_G$ is no R-implicator either.

Example 9 (*Aggregated implicators on L^**). In [9] Bustince et al. constructed implication operators for intuitionistic fuzzy logic based on aggregation operators on $[0,1]$. Recall that an aggregation operator is a $[0,1]^2 \to [0,1]$ mapping M that satisfies the following conditions:

(1) $M(0,0) = 0$
(2) $M(1,1) = 1$
(3) M is increasing in its first and in its second argument
(4) $M(x,y) = M(y,x)$ for all $x, y \in [0,1]$

They proved that if I is an implicator and N an involutive negator on $[0,1]$, and M_1, M_2, M_3, and M_4 are aggregation operators such that $M_1(x,y) + M_3(1-x, 1-y) \geqslant 1$ and $M_2(x,y) + M_4(1-x, 1-y) \leqslant 1$ for all $x, y \in [0,1]$, then $\mathscr{I}$ defined by, for all $x, y \in L^*$,

$$\begin{aligned}\mathscr{I}(x,y) = (&I(M_1(x_1, 1-x_2), M_2(y_1, 1-y_2)), N(I(N(M_3(x_2, 1-x_1)),\\ &N(M_4(y_2, 1-y_1)))))\end{aligned}$$

is an implicator on L^* in the sense of Definition 6.

As a simple instance of this class, putting $M_1 = M_3 = \max$, $M_2 = M_4 = \min$ and I the Kleene–Dienes implicator on $[0,1]$, we obtain the S-implicator from Example 2. More interesting implicators emerge when the aggregation operators are chosen strictly between min and max, i.e. $\min(x,y) < M_i(x,y) < \max(x,y)$, for some $x, y \in [0,1]$ and $i = 1, \ldots, 4$. For instance, putting $M_1 = M_2 = M_3 = M_4 = M$ with $M(x,y) = \frac{x+y}{2}$ for all $x, y \in [0,1]$, we obtain the following implicator $\mathscr{I}$ on L^*:

$$\mathscr{I}_B(x,y) = \left(\max\left(\frac{1-x_1+x_2}{2}, \frac{1-y_2+y_1}{2}\right), \min\left(\frac{1-x_2+x_1}{2}, \frac{1-y_1+y_2}{2}\right)\right)$$

This implicator has no representation in terms of S- nor R-implicators. Indeed, suppose $\mathscr{I}_B(x,y) = \mathscr{S}(\mathscr{N}(x), y)$ for all $x, y \in L^*$, $\mathscr{S}$ a t-conorm and $\mathscr{N}$ a negator on L^*. Put $x = 1_{L^*}, y = (\frac{1}{2}, \frac{1}{4})$. Then

$$\begin{aligned}\mathscr{I}_B(x,y) &= \left(\max\left(\frac{1-1+0}{2}, \frac{1-\frac{1}{4}+\frac{1}{2}}{2}\right), \min\left(\frac{1-0+1}{2}, \frac{1-\frac{1}{2}+\frac{1}{4}}{2}\right)\right)\\ &= \left(\frac{5}{8}, \frac{3}{8}\right)\end{aligned}$$

On the other hand, $\mathscr{S}(\mathscr{N}(x),y) = \mathscr{S}(0_{L^*},y) = y \neq (\frac{5}{8},\frac{3}{8})$, a contradiction, so $\mathscr{I}_B$ cannot be an S-implicator.

Suppose on the other hand that $\mathscr{I}_B(x,y) = \sup\{\gamma \in L^* | \mathscr{T}(x,\gamma) \leqslant_{L^*} y\}$ for all $x,y \in L^*$, and $\mathscr{T}$ a t-norm on L^*. Put $x = y = (\frac{1}{4},\frac{1}{4})$. Then $\mathscr{I}_B(x,y) = (\frac{1}{2},\frac{1}{2})$. But $\mathscr{I}_{\text{Min}}(x,y) = 1_{L^*}$, thus $\mathscr{I}_B(x,y) \ngeqslant_{L^*} \mathscr{I}_{\text{Min}}(x,y)$ and hence $\mathscr{I}_B$ has no representation as an R-implicator.

Example 10 (*Wu implicator on L^**). The mapping $\mathscr{I}_{wu}$ on L^* defined by, for $x,y \in L^*$:

$$\mathscr{I}_{wu}(x,y) = \begin{cases} 1_{L^*} & \text{if } x \leqslant_{L^*} y \\ \text{Min}(\mathscr{N}_s(x),y) & \text{otherwise} \end{cases}$$

is an implicator on L^*: if $x \leqslant_{L^*} x'$, then it follows easily that $\mathscr{I}_{wu}(x,y) \geqslant_{L^*} \mathscr{I}_{wu}(x',y)$, since $1_{L^*} \geqslant_{L^*} \text{Min}(\mathscr{N}_s(x),y)$, and $x \leqslant_{L^*} x'$ implies $\text{Min}(\mathscr{N}_s(x),y) \geqslant_{L^*} \text{Min}(\mathscr{N}_s(x'),y)$. If $y \leqslant_{L^*} y'$, then $\text{Min}(\mathscr{N}_s(x),y) \leqslant_{L^*} \text{Min}(\mathscr{N}_s(x),y') \leqslant_{L^*} 1_{L^*}$, from which follows easily that $\mathscr{I}_{wu}(x,y) \leqslant_{L^*} \mathscr{I}_{wu}(x,y')$.

$\mathscr{I}_{wu}$ is an extension of the implicator on $[0,1]$ introduced by Wu in [64]. Since that implicator is neither an S- nor an R-implicator, the Wu implicator on L^* likewise is not.

We conclude with an example of a mapping that was designated as an intuitionistic fuzzy implicator, but in fact does not meet the criteria of Definition 6.

Example 11. In [3], Atanassov and Gargov defined the following $(L^*)^2 \to L^*$ mapping J:

$$J(x,y) = \begin{cases} 1_{L^*} & \text{if } x \leqslant_{L^*} y \\ (y_1,x_2) & \text{if } x_1 > y_1 \quad \text{and} \quad x_2 \geqslant y_2 \\ (x_1,y_2) & \text{if } x_1 \leqslant y_1 \quad \text{and} \quad x_2 < y_2 \\ 0_{L^*} & \text{if } x >_{L^*} y \end{cases}$$

It is not decreasing in its first component. Indeed, put $x = (0.6, 0.2)$, $x' = (0.7, 0.15)$ and $y = (0.4, 0.1)$. Then $x \leqslant_{L^*} x'$, but $J(x,y) = (0.4, 0.2) <_{L^*} J(x',y) = (0.4, 0.15)$.

4. Classification of implicators on L^*: an algebraic approach

The task of classifying implicators defined within any many-valued extension of classical binary propositional logic essentially comes down to checking how many desirable properties of the original operation are kept by the extended structure. Therefore, regardless of background and goals, inspiration for this process draws primarily from logic, and the notion of a tautology (meaning a formula whose truth value is always 1_{L^*}, regardless of the truth values of its constituents), or a weakened version of it, [5] will play an important role in it.

Fuzzy logics, as examples of well-studied many-valued truth structures, have shown how an algebraic treatment—translating desirable properties into algebraic laws (or axioms) to be satisfied, and subsequently tracing the shape an implicator should have in order to satisfy a list of axioms—can shed a systematic and pragmatical light on the subject by providing a yardstick method for "measuring" the usefulness of implicators. We pursue this strategy for implicators on L^*, taking as our point of departure an extended version of the Smets–Magrez axioms and finishing off with an algebraic representation of implicators satisfying the entire axiom list. This strategy is therefore dubbed an "algebraic approach".

4.1. Axioms of Smets and Magrez

In [55], Smets and Magrez outlined an axiom scheme for implicators on $[0,1]$. They took a number of important tautologies from classical logic, translated them into four algebraic axioms and complemented the scheme with monotonicity (A.1) and continuity (A.6) requirements that emerge naturally when we swap the discrete space $\{0,1\}$ for the continuum $[0,1]$. Smets–Magrez axioms stand as a yardstick to test the suitability of implicators on $[0,1]$; it is therefore instructive to generalize them to L^*:

Definition 9 (*Smets–Magrez axioms for an implicator $\mathscr{I}$ on L^**)

(A.1) $(\forall y \in L^*)$ ($\mathscr{I}(.,y)$ is decreasing in L^*) and $(\forall x \in L^*)$ ($\mathscr{I}(x,.)$ is increasing in L^*) (monotonicity laws)
(A.2) $(\forall x \in L^*)$ $(\mathscr{I}(1_{L^*},x) = x)$ (neutrality principle)
(A.3) $\left(\forall (x,y) \in (L^*)^2\right)$ $(\mathscr{I}(x,y) = \mathscr{I}(\mathscr{N}_{\mathscr{I}}(y), \mathscr{N}_{\mathscr{I}}(x)))$ (contrapositivity)
(A.4) $\left(\forall (x,y,z) \in (L^*)^3\right)$ $(\mathscr{I}(x,\mathscr{I}(y,z)) = \mathscr{I}(y,\mathscr{I}(x,z)))$ (interchangeability principle)
(A.5) $\left(\forall (x,y) \in (L^*)^2\right)$ $(x \leqslant_{L^*} y \Longleftrightarrow \mathscr{I}(x,y) = 1_{L^*})$ (confinement principle)
(A.6) $\mathscr{I}$ is a continuous $(L^*)^2 \to L^*$ mapping (continuity).

[5] In concreto, we think about the following two possible variations on a tautology:

- Fuzzy tautologies [49], formulas whose truth values (x_1,x_2) are such that $x_1 \geqslant 0.5$.
- Intuitionistic fuzzy tautologies [3], formulas whose truth values (x_1,x_2) are such that $x_1 \geqslant x_2$.

Note. In axiom 3, the mapping $\mathscr{N}_{\mathscr{I}}$, defined by $\mathscr{N}_{\mathscr{I}}(x) = \mathscr{I}(x, 0_{L^*})$, is a negator on L^*. It is called the negator induced by $\mathscr{I}$. It can be easily verified that if (A.2) and (A.3) hold, then necessarily $\mathscr{N}_{\mathscr{I}}$ is involutive.

In what follows, we first conduct a detailed investigation to verify if and under which conditions the implicators of the previous section satisfy the various Smets–Magrez axioms. Afterwards, we proceed to derive an algebraic expression for general implicators on L^* satisfying the six axioms simultaneously. In this context, we need to introduce some additional terminology: an implicator in the sense of Definition 6 is called a *border implicator* if it satisfies the neutrality principle (A.2); a contrapositive border implicator additionally satisfying the interchangeability principle (A.4) is called a *model implicator*; and finally, a continuous model implicator for which (A.5) is also verified, is called a *Łukasiewicz implicator*.

4.2. Smets–Magrez axioms for S- and R-implicators on L^*

As can be seen from Definition 9, axiom 1 merely asserts the monotonicity conditions incorporated into the definition of implicators on L^*. It is kept in the list for reasons of tradition, but will not occur in the following discussion.

4.2.1. S-implicators

Theorem 4. *An S-implicator $\mathscr{I}_{\mathscr{S},\mathscr{N}}$ on L^* is a model implicator on the condition that $\mathscr{N}$ is involutive.*

Proof. We verify that each of the axioms (A.2), (A.3) and (A.4) is fulfilled. Let $x, y, z \in L^*$.

$$\text{(A.2)}\quad \mathscr{I}_{\mathscr{S},\mathscr{N}}(1_{L^*}, x) = \mathscr{S}(\mathscr{N}(1_{L^*}), x) = S(0_{L^*}, x) = x$$

$$\text{(A.3)}\quad \mathscr{I}_{\mathscr{S},\mathscr{N}}(\mathscr{I}_{\mathscr{S},\mathscr{N}}(y, 0_{L^*}), \mathscr{I}_{\mathscr{S},\mathscr{N}}(x, 0_{L^*})) = \mathscr{S}(\mathscr{N}(\mathscr{N}(y)), \mathscr{N}(x)) = \mathscr{S}(y, \mathscr{N}(x)) = \mathscr{S}(\mathscr{N}(x), y) = \mathscr{I}_{\mathscr{S},\mathscr{N}}(x, y)$$

$$\begin{aligned}\text{(A.4)}\quad \mathscr{I}_{\mathscr{S},\mathscr{N}}(x, \mathscr{I}_{\mathscr{S},\mathscr{N}}(y, z)) &= \mathscr{S}(\mathscr{N}(x), \mathscr{S}(\mathscr{N}(y), z)) \\ &= \mathscr{S}(\mathscr{S}(\mathscr{N}(x), \mathscr{N}(y)), z) \\ &= \mathscr{S}(\mathscr{S}(\mathscr{N}(y), \mathscr{N}(x)), z) \\ &= \mathscr{S}(\mathscr{N}(y), \mathscr{S}(\mathscr{N}(x), z)) \\ &= \mathscr{I}_{\mathscr{S},\mathscr{N}}(y, \mathscr{I}_{\mathscr{S},\mathscr{N}}(x, z))\end{aligned}$$

The deduction for axiom (A.3) requires the involutivity of $\mathscr{N}$. □

As the following theorem shows, t-representability presents us with an important obstacle in our quest for a Łukasiewicz implicator on L^*.

Theorem 5. *Axiom* (A.5) *fails for every t-representable S-implicator* $\mathscr{I}_{\mathscr{S},\mathscr{N}}$ *on* L^*, *provided that* $\mathscr{N}$ *is involutive.*

Proof. Assume that the representants of $\mathscr{S}$ are S and T. By definition we have, for $x, y \in L^*$:

$$\mathscr{I}_{\mathscr{S},\mathscr{N}}(x, y) = (S(\mathrm{pr}_1\mathscr{N}(x), y_1), T(\mathrm{pr}_2\mathscr{N}(x), y_2))$$

Let $y = (0, y_2) \in L^*$, $x = (0, x_2) \in L^*$ such that $1 > x_2 \geqslant y_2$, so $x \leqslant_{L^*} y$. Then

$$\begin{aligned}\mathscr{I}_{\mathscr{S},\mathscr{N}}(x, y) &= (S(\mathrm{pr}_1\mathscr{N}(x), 0), T(\mathrm{pr}_2\mathscr{N}(x), y_2)) \\ &= (\mathrm{pr}_1\mathscr{N}(x), T(\mathrm{pr}_2\mathscr{N}(x), y_2))\end{aligned}$$

If $\mathrm{pr}_1\mathscr{N}(x) = 1$ then $\mathrm{pr}_2\mathscr{N}(x) = 0$ and thus $x = \mathscr{N}(\mathscr{N}(x)) = \mathscr{N}(1_{L^*}) = 0_{L^*}$, which contradicts our assumptions about x. Hence $\mathrm{pr}_1\mathscr{N}(x) \neq 1$, and thus $\mathscr{I}_{\mathscr{S},\mathscr{N}}(x, y) \neq 1_{L^*}$. □

Theorem 6. *Axiom* (A.6) *holds for an S-implicator* $\mathscr{I}_{\mathscr{S},\mathscr{N}}$ *on* L^* *as soon as* $\mathscr{S}$ *and* $\mathscr{N}$ *are continuous. In particular, a t-representable S-implicator represented by T and S is continuous as soon as T, S and* $\mathscr{N}$ *are continuous.*

Proof. This is obvious by the chaining rule for continuous mappings on the subspace L^* of $\mathbb{R}^2$. □

4.2.2. R-implicators

Theorem 7. *An R-implicator* $\mathscr{I}_{\mathscr{T}}$ *on* L^* *is a border implicator.*

Proof. We only have to verify (A.2). Let $x \in L^*$. We have:

$$\sup\{\gamma \in L^* | \mathscr{T}(1_{L^*}, \gamma) \leqslant_{L^*} x\} = \sup\{\gamma \in L^* | \gamma \leqslant_{L^*} x\} = x \qquad \square$$

Again, problems emerge w.r.t. t-representability, this time concerning the contrapositivity of the implicator.

Theorem 8. *Axiom* (A.3) *does not hold for any t-representable R-implicator* $\mathscr{I}_{\mathscr{T}}$ *on* L^*.

Proof. Assume that the representants of $\mathscr{T}$ are T and S. Let $x, y \in L^*$, and suppose (A.3) holds. Then $\mathscr{N}_{\mathscr{I}_{\mathscr{T}}}$ is involutive. We have:

$$\mathscr{I}_{\mathscr{T}}(\mathscr{N}_{\mathscr{I}_{\mathscr{T}}}(y), \mathscr{N}_{\mathscr{I}_{\mathscr{T}}}(x)) = \sup\{\gamma \in L^* | \mathscr{T}(\mathscr{N}_{\mathscr{I}_{\mathscr{T}}}(y), \gamma) \leqslant_{L^*} \mathscr{N}_{\mathscr{I}_{\mathscr{T}}}(x)\}$$

Let $y = 0_{L^*}$, then $\mathcal{N}_{\mathcal{I}_{\mathcal{T}}}(y) = 1_{L^*}$ and

$$\mathcal{I}_{\mathcal{T}}(1_{L^*}, \mathcal{N}_{\mathcal{I}_{\mathcal{T}}}(x)) = \mathcal{N}_{\mathcal{I}_{\mathcal{T}}}(x)$$

We also have $\mathcal{I}_{\mathcal{T}}(x, 0_{L^*}) = \mathcal{N}_{\mathcal{I}_{\mathcal{T}}}(x)$, in other words:

$$\sup\{(\gamma_1, \gamma_2) \in L^* | (\mathcal{T}(x_1, \gamma_1), S(x_2, \gamma_2)) \leqslant_{L^*} 0_{L^*}\} = \mathcal{N}_{\mathcal{I}_{\mathcal{T}}}(x)$$

Let $x = (x_1, 0) \neq 1_{L^*}$, then $\mathcal{N}_{\mathcal{I}_{\mathcal{T}}}(x) \neq 0_{L^*}$ since $\mathcal{N}_{\mathcal{I}_{\mathcal{T}}}$ is involutive. We find:

$$\mathcal{I}_{\mathcal{T}}(x, 0_{L^*}) = \sup\{(\gamma_1, \gamma_2) \in L^* | (T(x_1, \gamma_1), \gamma_2) \leqslant_{L^*} 0_{L^*}\} = \mathcal{N}_{\mathcal{I}_{\mathcal{T}}}(x)$$

Since $\inf\{\gamma_2 \in [0, 1] | \gamma_2 \geqslant 1\} = 1$, we obtain

$$\mathcal{I}_{\mathcal{T}}(x, 0_{L^*}) = 0_{L^*} \neq \mathcal{N}_{\mathcal{I}_{\mathcal{T}}}(x)$$

which is a contradiction. In other words, (A.3) does not hold. □

Theorem 9. *Axiom (A.5) holds for the R-implicator $\mathcal{I}_{\mathcal{T}}$ if and only if there exists for each $x = (x_1, x_2) \in L^*$ a sequence $(\delta_i)_{i \in \mathbb{N}^*}$ in $\Omega = \{\delta \in L^* | \delta_2 > 0\}$ such that $\lim_{i \to \infty} \delta_i = 1_{L^*}$ and,*

$$\lim_{\delta_i \to 1_{L^*}} \mathrm{pr}_1 \mathcal{T}(x, \delta_i) = x_1 \quad (2)$$

$$\lim_{\delta_i \to 1_{L^*}} \mathrm{pr}_2 \mathcal{T}(x, \delta_i) = x_2 \quad (3)$$

Proof. Assume first that conditions (2) and (3) are fulfilled. We start by proving that

$$(\forall (x, y) \in (L^*)^2)(x \leqslant_{L^*} y \Rightarrow \mathcal{I}_{\mathcal{T}}(x, y) = 1_{L^*})$$

Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in L^*$ such that $x \leqslant_{L^*} y$, then $\forall \gamma \in L^*$, $\mathcal{T}(x, \gamma) \leqslant_{L^*} x \leqslant_{L^*} y$. Hence

$$\sup\{\gamma \in L^* | \mathcal{T}(x, \gamma) \leqslant_{L^*} y\} = 1_{L^*}$$

To prove the converse implication,

$$\left(\forall (x, y) \in (L^*)^2\right)(x \leqslant_{L^*} y \Leftarrow \mathcal{I}_{\mathcal{T}}(x, y) = 1_{L^*}),$$

let $x = (x_1, x_2)$, $y = (y_1, y_2) \in L^*$. From

$$\mathcal{I}_{\mathcal{T}}(x, y) = \sup\{\gamma \in L^* | \mathcal{T}(x, \gamma) \leqslant_{L^*} y\} = 1_{L^*}$$

it follows that $\Omega \subseteq \{\gamma \in L^* | \mathcal{T}(x, \gamma) \leqslant_{L^*} y\}$, and so $\mathrm{pr}_1 \mathcal{T}(x, \delta_i) \leqslant y_1$ $\forall i \in \mathbb{N}^*$, so $\lim_{\delta_i \to 1_{L^*}} \mathrm{pr}_1 \mathcal{T}(x, \delta_i) = x_1 \leqslant y_1$. Similarly we obtain $\lim_{\delta_i \to 1_{L^*}} \mathrm{pr}_2 \mathcal{T}(x, \delta_i) = x_2 \geqslant y_2$. Hence $x \leqslant_{L^*} y$.

Conversely, assume that (A.5) holds. Suppose now that for each sequence $(\delta_i)_{i \in \mathbb{N}^*}$ in Ω converging to 1_{L^*}, either $\lim_{\delta_i \to 1_{L^*}} \mathrm{pr}_1 \mathcal{T}(x, \delta_i)$ is strictly smaller than x_1, or does not exist, or that $\lim_{\delta_i \to 1_{L^*}} \mathrm{pr}_2 \mathcal{T}(x, \delta_i)$ is strictly greater than x_2 or does not exist.

Let now

$$y = \sup\{\mathscr{T}(x,\gamma)|\gamma \in \Omega\},$$

then since $\mathscr{T}(x,\gamma) \leqslant_{L^*} x$ for all $\gamma \in L^*$, $y \leqslant_{L^*} x$. Suppose $y = x$. Let then $\epsilon = \frac{1}{n}$, with $n \in \mathbb{N}^*$. Then, since $x_1 = \sup\{\mathrm{pr}_1\mathscr{T}(x,\gamma)|\gamma \in \Omega\}$, there exists a $\gamma_n \in \Omega$ such that $x_1 - \epsilon < \mathrm{pr}_1\mathscr{T}(x,\gamma_n) \leqslant x_1$, thus $|x_1 - \mathrm{pr}_1\mathscr{T}(x,\gamma_n)| < \epsilon = \frac{1}{n}$. Similarly, there exists a $\gamma'_n \in \Omega$ such that $|x_2 - \mathrm{pr}_2\mathscr{T}(x,\gamma'_n)| < \epsilon$. Let now $\gamma''_n = \sup\{\gamma_n, \gamma'_n, (1 - \frac{1}{n}, \frac{1}{n})\}$. Then $\gamma''_n \geqslant_{L^*} \gamma_n$, so $\mathrm{pr}_1\mathscr{T}(x,\gamma''_n) \geqslant \mathrm{pr}_1\mathscr{T}(x,\gamma_n)$, and similarly $\mathrm{pr}_2\mathscr{T}(x,\gamma''_n) \leqslant \mathrm{pr}_2\mathscr{T}(x,\gamma'_n)$. Furthermore $\gamma''_n \in \Omega$, since $\gamma''_{n,2} = \min\{\gamma_{n,2}, \gamma'_{n,2}, \frac{1}{n}\} > 0$. Thus we obtain a sequence $(\gamma''_n)_{n\in\mathbb{N}^*}$ in Ω such that $|\gamma''_{n,1} - 1| + |\gamma''_{n,2} - 0| \leqslant \frac{2}{n}$ and $|\mathrm{pr}_1\mathscr{T}(x,\gamma''_n) - x_1| + |\mathrm{pr}_2\mathscr{T}(x,\gamma''_n) - x_2| < \frac{2}{n}$. Clearly $\lim_{n\to+\infty}\gamma''_n = 1_{L^*}$ and $\lim_{n\to+\infty}\mathscr{T}(x,\gamma''_n) = x$, which is in contradiction with our assumption. Hence $y <_{L^*} x$, so $x \not\leqslant_{L^*} y$. Clearly

$$\sup\{\gamma \in L^*|\mathscr{T}(x,\gamma) \leqslant_{L^*} y\} = 1_{L^*}$$

In other words, (A.5) is violated, so our assumption that conditions (2) and (3) do not hold was false. □

Corollary 10. *If $\mathscr{T}$ is a t-norm on L^* such that $\mathrm{pr}_1\mathscr{T}$ is a left-continuous $(L^*)^2 \to [0,1]$ mapping and $\mathrm{pr}_2\mathscr{T}$ is a right-continuous $(L^*)^2 \to [0,1]$ mapping then $\mathscr{I}_{\mathscr{T}}$ satisfies* (A.5).

Proof. We will prove this result by constructing a sequence meeting the conditions of Theorem 9. From the left-continuity of $\mathrm{pr}_1\mathscr{T}$ and the right-continuity of $\mathrm{pr}_2\mathscr{T}$ follows for all $x, y \in L^*$ [29]:

$$\begin{aligned}&(\forall\epsilon > 0)(\exists\delta_1 > 0)(\exists\delta_2 > 0)(\forall y' \in L^*)(y_1 - \delta_1 < y'_1 \leqslant y_1 \quad \text{and}\\ &y_2 \leqslant y'_2 < y_2 + \delta_2 \Rightarrow |\mathrm{pr}_1\mathscr{T}(x,y) - \mathrm{pr}_1\mathscr{T}(x,y')| + |\mathrm{pr}_2\mathscr{T}(x,y)\\ &\qquad - \mathrm{pr}_2\mathscr{T}(x,y')| < \epsilon)\end{aligned}$$

In [29] this property is proven to be equivalent to

$$\mathscr{T}(x, \sup A) = \sup_{y\in A}\mathscr{T}(x,y)$$

for any $x \in L^*$ and any subset A of L^*. So let arbitrarily $x \in L^*$. Then we obtain $\sup_{\gamma\in\Omega}\mathscr{T}(x,\gamma) = \mathscr{T}(x, \sup\Omega)$, i.e. $\sup\{\mathscr{T}(x,\gamma)|\gamma \in \Omega\} = \mathscr{T}(x, 1_{L^*}) = x$, where $\Omega = \{\delta \in L^*|\delta_2 > 0\}$. Similarly as in Theorem 9 a sequence can be constructed which satisfies the desired properties. □

To convince the reader that *t*-representability really does impose an unacceptable restriction on an implicator on L^*, we now prove that $\mathscr{I}_{\mathscr{T}_W} = \mathscr{I}_{\mathscr{S}_W,\mathscr{N}_s}$, i.e. the implicator derived in Examples 4 and 7, satisfies all Smets–Magrez axioms, showing at the same time that a Łukasiewicz implicator on L^* can be both an S- and an R-implicator.

Theorem 11. *$\mathscr{I}_{\mathscr{S}_W,\mathscr{N}_s}$ is a Łukasiewicz implicator.*

Proof. $\mathscr{I}_{\mathscr{S}_W,\mathscr{N}_s}$ satisfies (A.1)–(A.4) because it is an S-implicator on L^* and $\mathscr{N}_s$ is involutive. Since $\mathscr{S}_W$ and $\mathscr{N}_s$ are continuous, so is $\mathscr{I}_{\mathscr{S}_W,\mathscr{N}_s}$. Only (A.5) is left to verify. Recall the definition of $\mathscr{I}_{\mathscr{S}_W,\mathscr{N}_s}$, i.e. for all $x, y \in L^*$:

$$\mathscr{I}_{\mathscr{S}_W,\mathscr{N}_s}(x,y) = (\min(1, y_1 + 1 - x_1, x_2 + 1 - y_2), \max(0, y_2 + x_1 - 1))$$

We find: $y_1 + 1 - x_1 < 1$ iff $y_1 < x_1$, $x_2 + 1 - y_2 < 1$ iff $x_2 < y_2$. Hence $\min(1, x_2 + 1 - y_2, y_1 + 1 - x_1) < 1$ iff either $y_1 < x_1$ or $x_2 < y_2$. So $\min(1, x_2 + 1 - y_2, y_1 + 1 - x_1) = 1$ iff $y_1 \geqslant x_1$ and $x_2 \geqslant y_2$, i.e. iff $x \leqslant_{L^*} y$. □

In Table 1, we have summarized the classification results w.r.t. the extended Smets–Magrez axioms. For completeness, apart from S- and R-implicators, we have also included the implicators discussed in Section 3.3. It is left to the reader to verify these properties.

A question unanswered by this table is whether there exist Łukasiewicz implicators on L^* outside the classes of S- and R-implicators. This and other issues are resolved in the following paragraph.

4.3. Representation of model and Łukasiewicz implicators on L^*

We have shown, by explicit example, that a Łukasiewicz implicator on L^* *exists*. The next question to ask is whether we can capture *all* of them by a parameterized formula, as was done for implicators on [0, 1] (see e.g. [46]). The answer turns out to be largely affirmative, as the following discussion reveals.

Table 1
Smets–Magrez axioms for a number of implicators and implicator classes on L^*

	(A.1)	(A.2)	(A.3)	(A.4)	(A.5)	(A.6)
S-implicators	yes	yes	provided $\mathscr{N}$ involutive	yes	e.g. $\mathscr{I}_{\mathscr{T}_W}$	provided $\mathscr{S}$ and $\mathscr{N}$ continuous
t-representable S-implicators	yes	yes	provided $\mathscr{N}$ involutive	yes	no	provided S, T and $\mathscr{N}$ continuous
R-implicators	yes	yes	e.g. $\mathscr{I}_{\mathscr{T}_W}$	e.g. $\mathscr{I}_{\mathscr{T}_W}$	Theorem 9	e.g. $\mathscr{I}_{\mathscr{T}_W}$
t-representable R-implicators	yes	yes	no	unknown	e.g. Example 5	unknown
$\mathscr{I}_{\text{ag}}$	yes	yes	no	yes	no	no
$\mathscr{I}_{\text{g}}$	yes	yes	no	yes	yes	no
$\mathscr{I}_B$	yes	no	yes	yes	no	yes
$\mathscr{I}_{wu}$	yes	no	no	no	yes	no
J	no	no	no	no	yes	no

A first important subresult in this direction is the observation that each model implicator on L^* has a representation as an S-implicator.

Lemma 1 (Triangular norm and conorm induced by a model implicator). *If $\mathscr{I}$ is a model implicator, then the $(L^*)^2 \to L^*$-mappings $\mathscr{T}_{\mathscr{I}}$ and $\mathscr{S}_{\mathscr{I}}$ defined by, for $x, y \in L^*$,*

$$\mathscr{T}_{\mathscr{I}}(x,y) = \mathscr{N}_{\mathscr{I}}(\mathscr{I}(x, \mathscr{N}_{\mathscr{I}}(y)))$$
$$\mathscr{S}_{\mathscr{I}}(x,y) = \mathscr{I}(\mathscr{N}_{\mathscr{I}}(x), y)$$

are a t-norm and an t-conorm on L^*, respectively. They are called the t-norm and t-conorm induced by $\mathscr{I}$.

Proof. We prove the claim for $\mathscr{T}_{\mathscr{I}}$. The proof for $\mathscr{S}_{\mathscr{I}}$ is analogous.

- $\mathscr{T}_{\mathscr{I}}$ is increasing. This is obvious because $\mathscr{I}$ is an implicator and $\mathscr{N}$ a negator on L^*.
- $\mathscr{T}_{\mathscr{I}}$ is commutative. Indeed, for $x, y \in L^*$, we have:

$$\begin{aligned}
\mathscr{T}_{\mathscr{I}}(x,y) &= \mathscr{N}_{\mathscr{I}}(\mathscr{I}(x, \mathscr{N}_{\mathscr{I}}(y))) && \text{Definition } \mathscr{T}_{\mathscr{I}} \\
&= \mathscr{N}_{\mathscr{I}}(\mathscr{I}(\mathscr{N}_{\mathscr{I}}(\mathscr{N}_{\mathscr{I}}(y)), \mathscr{N}_{\mathscr{I}}(x))) && \mathscr{I} \text{ contrapositive w.r.t. } \mathscr{N}_{\mathscr{I}} \\
&= \mathscr{N}_{\mathscr{I}}(\mathscr{I}(y, \mathscr{N}_{\mathscr{I}}(x))) && \mathscr{N}_{\mathscr{I}} \text{ is involutive} \\
&= \mathscr{T}_{\mathscr{I}}(y,x) && \text{Definition } \mathscr{T}_{\mathscr{I}}
\end{aligned}$$

- $\mathscr{T}_{\mathscr{I}}$ is associative. Indeed, for $x, y, z \in L^*$, we have:

$$\begin{aligned}
&\mathscr{T}_{\mathscr{I}}(x, \mathscr{T}_{\mathscr{I}}(y,z)) \\
&\quad = \mathscr{N}_{\mathscr{I}}(\mathscr{I}(x, \mathscr{N}_{\mathscr{I}}(\mathscr{N}_{\mathscr{I}}(\mathscr{I}(y, \mathscr{N}_{I}(z)))))) && \text{Definition } \mathscr{T}_{\mathscr{I}} \\
&\quad = \mathscr{N}_{\mathscr{I}}(\mathscr{I}(x, \mathscr{I}(y, \mathscr{N}_{\mathscr{I}}(z)))) && \mathscr{N}_{\mathscr{I}} \text{ is involutive} \\
&\quad = \mathscr{N}_{\mathscr{I}}(\mathscr{I}(x, \mathscr{I}(\mathscr{N}_{\mathscr{I}}(\mathscr{N}_{\mathscr{I}}(z)), \mathscr{N}_{\mathscr{I}}(y)))) && \mathscr{I} \text{ is contrapositive w.r.t. } \mathscr{N}_{\mathscr{I}} \\
&\quad = \mathscr{N}_{\mathscr{I}}(\mathscr{I}(x, \mathscr{I}(z, \mathscr{N}_{\mathscr{I}}(y)))) && \mathscr{N}_{\mathscr{I}} \text{ is involutive} \\
&\quad = \mathscr{N}_{\mathscr{I}}(\mathscr{I}(z, \mathscr{I}(x, \mathscr{N}_{\mathscr{I}}(y)))) && \mathscr{I} \text{ satisfies (A.4)} \\
&\quad = \mathscr{N}_{\mathscr{I}}(\mathscr{I}(\mathscr{N}_{\mathscr{I}}(\mathscr{I}(x, \mathscr{N}_{\mathscr{I}}(y))), \mathscr{N}_{\mathscr{I}}(z))) && \mathscr{I} \text{ is contrapositive w.r.t. } \mathscr{N}_{\mathscr{I}} \\
&\quad = \mathscr{T}_{\mathscr{I}}(\mathscr{T}_{\mathscr{I}}(x,y), z) && \text{Definition } \mathscr{T}_{\mathscr{I}}
\end{aligned}$$

- $\mathscr{T}_{\mathscr{I}}(1_{L^*}, x) = x$. Indeed, for $x, y \in L^*$, we have:

$$\begin{aligned}
\mathscr{T}_{\mathscr{I}}(1_{L^*}, x) &= \mathscr{N}_{\mathscr{I}}(\mathscr{I}(1_{L^*}, \mathscr{N}_{\mathscr{I}}(x))) && \text{Definition } \mathscr{T}_{\mathscr{I}} \\
&= \mathscr{N}_{\mathscr{I}}(\mathscr{N}_{\mathscr{I}}(x)) && \mathscr{I} \text{ satisfies (A.2)} \\
&= x && \mathscr{N}_{\mathscr{I}} \text{ is involutive} \quad \square
\end{aligned}$$

Definition 10 (*IF de Morgan triplet*). An IF de Morgan triplet is any triplet $(\mathscr{T}, \mathscr{S}, \mathscr{N})$ consisting of a *t*-norm $\mathscr{T}$, a *t*-conorm $\mathscr{S}$ and a negator $\mathscr{N}$ on L^* such that, for all $x, y \in L^*$:

$$\mathscr{N}(\mathscr{T}(\mathscr{N}(x),\mathscr{N}(y))) = \mathscr{S}(x,y)$$
$$\mathscr{N}(\mathscr{S}(\mathscr{N}(x),\mathscr{N}(y))) = \mathscr{T}(x,y)$$

Lemma 2. *If $\mathscr{I}$ is a model implicator on L^*, then $(\mathscr{T}_{\mathscr{I}},\mathscr{S}_{\mathscr{I}},\mathscr{N}_{\mathscr{I}})$ is an IF de Morgan triplet.*

Proof. We have to prove that $\mathscr{T}_{\mathscr{I}}$ and $\mathscr{S}_{\mathscr{I}}$ are dual w.r.t $\mathscr{N}_{\mathscr{I}}$. For $x,y \in L^*$, we have:

- $\mathscr{N}_{\mathscr{I}}(\mathscr{T}_{\mathscr{I}}(\mathscr{N}_{\mathscr{I}}(x),\mathscr{N}_{\mathscr{I}}(y)))$
  $= \mathscr{N}_{\mathscr{I}}(\mathscr{N}_{\mathscr{I}}(\mathscr{I}(\mathscr{N}_{\mathscr{I}}(x),\mathscr{N}_{\mathscr{I}}(\mathscr{N}_{\mathscr{I}}(y)))))$ Definition $\mathscr{T}_{\mathscr{I}}$
  $= \mathscr{I}(\mathscr{N}_{\mathscr{I}}(x),y)$ $\mathscr{N}_{\mathscr{I}}$ is involutive
  $= \mathscr{S}_{\mathscr{I}}(x,y)$ Definition $\mathscr{S}_{\mathscr{I}}$
- $\mathscr{N}_{\mathscr{I}}(\mathscr{S}_{\mathscr{I}}(\mathscr{N}_{\mathscr{I}}(x),\mathscr{N}_{\mathscr{I}}(y)))$
  $= \mathscr{N}_{\mathscr{I}}(\mathscr{I}(\mathscr{N}_{\mathscr{I}}(\mathscr{N}_{\mathscr{I}}(x)),\mathscr{N}_{\mathscr{I}}(y)))$ Definition $\mathscr{S}_{\mathscr{I}}$
  $= \mathscr{N}_{\mathscr{I}}(\mathscr{I}(x,\mathscr{N}_{\mathscr{I}}(y)))$ $\mathscr{N}_{\mathscr{I}}$ is involutive
  $= \mathscr{T}_{\mathscr{I}}(x,y)$ Definition $\mathscr{T}_{\mathscr{I}}$ □

Definition 11 (*IF de Morgan quartet*). An IF de Morgan quartet is any quartet $(\mathscr{T},\mathscr{S},\mathscr{N},\mathscr{I})$ consisting of a t-norm $\mathscr{T}$, a t-conorm $\mathscr{S}$, a negator $\mathscr{N}$ and an implicator $\mathscr{I}$ on L^* such that $(\mathscr{T},\mathscr{S},\mathscr{N})$ is an IF de Morgan triplet and, for all $x,y \in L^*$:

$$\mathscr{I}(x,y) = \mathscr{S}(\mathscr{N}(x),y)$$

Theorem 12. *If $\mathscr{I}$ is a model implicator on L^*, then $(\mathscr{T}_{\mathscr{I}},\mathscr{S}_{\mathscr{I}},\mathscr{N}_{\mathscr{I}},\mathscr{I})$ is an IF de Morgan quartet.*

Proof. From Lemma 2 we know that $(\mathscr{T}_{\mathscr{I}},\mathscr{S}_{\mathscr{I}},\mathscr{N}_{\mathscr{I}})$ is an IF de Morgan triplet. Furthermore, for all $x,y \in L^*$ we find:

$$\begin{aligned}\mathscr{S}_{\mathscr{I}}(\mathscr{N}_{\mathscr{I}}(x),y) &= \mathscr{I}(\mathscr{N}_{\mathscr{I}}(\mathscr{N}_{\mathscr{I}}(x)),y) && \text{Definition}\mathscr{S}_{\mathscr{I}}\\ &= \mathscr{I}(x,y) && \mathscr{N}_{\mathscr{I}} \text{ is involutive}\end{aligned}$$ □

Corollary 13. *A model implicator on L^* is an S-implicator.*

Proof. Indeed, from Theorem 12 we know that for a model implicator $\mathscr{I}$, $(\mathscr{T}_{\mathscr{I}},\mathscr{S}_{\mathscr{I}},\mathscr{N}_{\mathscr{I}},\mathscr{I})$ is a de Morgan quartet. Choose $\mathscr{S} = \mathscr{S}_{\mathscr{I}}$ and $\mathscr{N} = \mathscr{N}_{\mathscr{I}}$, then $\mathscr{I} = \mathscr{I}_{\mathscr{S},\mathscr{N}}$. □

This, and the results from the previous subsection, confirm that our search for Łukasiewicz implicators on L^* is limited to non-t-representable S-implicators.

We now proceed to establish a link between Łukasiewicz implicators on L^* and R-implicators generated by t-norms satisfying the conditions of Theorem 3. To this aim, a number of lemmas are introduced.

Theorem 14. *Let $\mathscr{T}$ be a t-norm on L^* satisfying the residuation principle. Then the following statements are equivalent:*

(i) $\mathscr{I}_{\mathscr{T}}$ *satisfies* (A.3)
(ii) $\mathscr{I}_{\mathscr{T}}(x, y) = \mathscr{N}(\mathscr{T}(x, \mathscr{N}(y)))$ *for all* $x, y \in L^*$
(iii) $\mathscr{T}(x, y) \leqslant_{L^*} z \iff \mathscr{T}(x, \mathscr{N}(z)) \leqslant_{L^*} \mathscr{N}(y)$ *for all* $x, y, z \in L^*$ (*exchange principle*)

where $\mathscr{N} = \mathscr{N}_{\mathscr{I}_{\mathscr{T}}}$. *Moreover, if* $\mathscr{T}$ *satisfies* (*iii*), *then* $\mathscr{T}$ *satisfies the residuation principle.*

Proof. We will prove that (iii) ⇒ (ii) ⇒ (i) ⇒ (iii). First note that since $\mathscr{I}_{\mathscr{T}}$ satisfies (A.2) and (A.3), $\mathscr{N}_{\mathscr{I}_{\mathscr{T}}}$ is involutive.

- Assume (iii) holds. The following deduction, for all $x, y, z \in L^*$, shows that (ii) holds.

$$\begin{aligned}\mathscr{I}_{\mathscr{T}}(x, y) &= \sup\{\gamma \in L^* | \mathscr{T}(x, \gamma) \leqslant_{L^*} y\}\\ &= \sup\{\gamma \in L^* | \mathscr{T}(x, \mathscr{N}(y)) \leqslant_{L^*} \mathscr{N}(\gamma)\}\\ &= \sup\{\gamma \in L^* | \gamma \leqslant_{L^*} \mathscr{N}(\mathscr{T}(x, \mathscr{N}(y)))\}\\ &= \mathscr{N}(\mathscr{T}(x, \mathscr{N}(y)))\end{aligned}$$

- Assume next that (ii) holds, then for all $x, y, z \in L^*$, we have:

$$\begin{aligned}\mathscr{I}_{\mathscr{T}}(\mathscr{N}(y), \mathscr{N}(x)) &= \mathscr{N}(\mathscr{T}(\mathscr{N}(y), \mathscr{N}(\mathscr{N}(x))))\\ &= \mathscr{N}(\mathscr{T}(\mathscr{N}(y), x))\\ &= \mathscr{N}(\mathscr{T}(x, \mathscr{N}(y)))\\ &= \mathscr{I}_{\mathscr{T}}(x, y)\end{aligned}$$

- Lastly, assume $\mathscr{I}_{\mathscr{T}}$ is contrapositive; we prove (iii). Since $\mathscr{I}_{\mathscr{T}}$ satisfies the residuation principle, we obtain successively, for all $x, y, z \in L^*$:

$$\begin{aligned}\mathscr{T}(x, y) \leqslant_{L^*} z &\iff \mathscr{T}(y, x) \leqslant_{L^*} z\\ &\iff x \leqslant_{L^*} \mathscr{I}_{\mathscr{T}}(y, z)\\ &\iff x \leqslant_{L^*} \mathscr{I}_{\mathscr{T}}(\mathscr{N}(z), \mathscr{N}(y))\\ &\iff \mathscr{T}(\mathscr{N}(z), x) \leqslant_{L^*} \mathscr{N}(y)\\ &\iff \mathscr{T}(x, \mathscr{N}(z)) \leqslant_{L^*} \mathscr{N}(y)\end{aligned}$$

Since from (iii) follows (ii), and using the fact that $\mathscr{N}$ is involutive and decreasing, we obtain successively:

$$\begin{aligned}\mathcal{T}(x,z) \leqslant_{L^*} y &\Longleftrightarrow \mathcal{T}(x,\mathcal{N}(y)) \leqslant_{L^*} \mathcal{N}(z)\\ &\Longleftrightarrow z \leqslant_{L^*} \mathcal{N}(\mathcal{T}(x,\mathcal{N}y))\\ &\Longleftrightarrow z \leqslant_{L^*} \mathcal{I}_{\mathcal{T}}(x,y)\end{aligned}$$

Hence from (iii) follows the residuation principle. □

Lemma 3. *Let $\mathcal{T}$ be a t-norm on L^* satisfying the exchange principle. Then $\mathcal{T}(x,y) = 0_{L^*} \Longleftrightarrow x \leqslant_{L^*} \mathcal{N}_{\mathcal{I}_{\mathcal{T}}}(y) \Longleftrightarrow y \leqslant_{L^*} \mathcal{N}_{\mathcal{I}_{\mathcal{T}}}(x)$.*

Proof. From the exchange principle follows $\mathcal{T}(x,y) = 0_{L^*} \Longleftrightarrow x = \mathcal{T}(x, 1_{L^*}) \leqslant_{L^*} \mathcal{N}_{\mathcal{I}_{\mathcal{T}}}(y)$. □

Lemma 4 [29]. *Let $\mathcal{T}$ be a t-norm on L^* satisfying the residuation principle. Then, for any x,y,z such that $\mathcal{T}(x,y) = z$, there exists an $y' \in L^*$ such that $y' \geqslant_{L^*} y$ and*

$$\mathcal{T}(x,y') = z \quad \textit{and} \quad y' = \mathcal{I}_{\mathcal{T}}(x,z). \tag{4}$$

Lemma 5. *Let $\mathcal{T}$ be a continuous t-norm on L^* satisfying $\mathcal{T}(D,D) \subseteq D$ and the exchange principle. Then $\mathcal{T}$ also satisfies the archimedean property, strong nilpotency, $\mathcal{I}_{\mathcal{T}}(D,D) \subseteq D$ and $\mathcal{T}((0,0),(0,0)) = 0_{L^*}$.*

Proof

- $\mathcal{T}((0,0),(0,0) = 0_{L^*}$.

  Since $\mathcal{N}_{\mathcal{I}_{\mathcal{T}}}$ is an involutive negator, we have that $\mathcal{N}_{\mathcal{I}_{\mathcal{T}}}(0,0) = (0,0)$ (see [29]). Hence $\mathcal{I}_{\mathcal{T}}((0,0),0_{L^*}) = (0,0)$ and from the residuation principle and Theorem 14 follows that $\mathcal{T}((0,0),(0,0)) = 0_{L^*}$.
- $\mathcal{T}$ is archimedean.

  Assume $x \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$ and $\mathcal{T}(x,x) = x$. Then, since $\mathcal{T}$ is increasing and $\mathcal{T}(x, 1_{L^*}) = x$, we obtain $\mathcal{T}(x,y) = x$ for all $y \geqslant_{L^*} x$. In particular $\mathcal{T}(x,(x_1,0)) = x$.

  If $x_1 = 0$, then $\mathcal{T}((0,x_2),(0,x_2)) \leqslant_{L^*} \mathcal{T}((0,0),(0,0)) = 0_{L^*} <_{L^*} (0,x_2)$. Suppose now $x_1 > 0$. We prove that there exists a sequence $(y_n)_{n\in\mathbb{N}^*}$ which converges to $(x_1,0)$ and such that, for all $n \in \mathbb{N}^*$, $y_n = (y_{n,1},0)$ and y_n satisfies

$$\mathcal{T}(x,\mathcal{N}_{\mathcal{I}_{\mathcal{T}}}(z_n)) = \mathcal{N}_{\mathcal{I}_{\mathcal{T}}}(y_n), \text{where } z_n = \mathcal{T}(x,y_n) \tag{5}$$

Let $n \in \mathbb{N}^*$. Since $\mathcal{T}(x,y) \leqslant_{L^*} y$ for all $y \in L^*$, we obtain $\mathrm{pr}_1\mathcal{T}(x,(x_1 - \frac{1}{n},0)) \leqslant x_1 - \frac{1}{n} < x_1$. Since $\mathcal{T}$ is increasing, we have $\mathcal{T}(x,(x_1 - \frac{1}{n},0)) \leqslant_{L^*} \mathcal{T}(x,1_{L^*}) = x$, so $\mathrm{pr}_2\mathcal{T}(x,(x_1 - \frac{1}{n},0)) \geqslant x_2$. Hence we obtain $\mathcal{T}(x,(x_1 - \frac{1}{n},0)) <_{L^*} x$. From Lemma 4 and Theorem 14 it follows that there exists a y_n such that $\mathcal{T}(x,y_n) = \mathcal{T}(x,(x_1 - \frac{1}{n},0))$ and y_n satisfies (5). Furthermore, from that lemma follows that $y_n \geqslant_{L^*} (x_1 - \frac{1}{n},0)$, so $x_1 - \frac{1}{n} \leqslant y_{n,1}$ and $y_{n,2} = 0$. Since $\mathcal{T}(x,y_n) <_{L^*} x$, it follows that $y_n \not\geqslant_{L^*} x$, thus $y_{n,1} < x_1$ (since $y_{n,2} = 0 \leqslant x_2$). From this follows that $|x_1 - y_{n,1}| + |0 - y_{n,2}| = x_1 - y_{n,1} \leqslant x_1 - (x_1 - \frac{1}{n}) = \frac{1}{n}$; hence

$\lim_{n\to+\infty} y_n = (x_1, 0)$. Moreover, since $\mathcal{T}$ is continuous, we have $\lim_{n\to+\infty} z_n = \lim_{n\to+\infty} \mathcal{T}(x, y_n) = \mathcal{T}(x, (x_1, 0)) = x$. From the involutivity (and hence the continuity) of $\mathcal{N}$ follows that $\lim_{n\to+\infty} \mathcal{N}_{\mathcal{I}_\mathcal{T}}(z_n) = \mathcal{N}_{\mathcal{I}_\mathcal{T}}(x)$, and $\lim_{n\to+\infty} \mathcal{N}_{\mathcal{I}_\mathcal{T}}(y_n) = \mathcal{N}_{\mathcal{I}_\mathcal{T}}(x_1, 0)$. Since y_n satisfies (5), we obtain $\mathcal{N}_{\mathcal{I}_\mathcal{T}}(x_1, 0) = \lim_{n\to+\infty} \mathcal{N}(y_n) = \lim_{n\to+\infty} \mathcal{T}(x, \mathcal{N}_{\mathcal{I}_\mathcal{T}}(z_n)) = \mathcal{T}(x, \mathcal{N}_{\mathcal{I}_\mathcal{T}}(x)) = 0_{L^*}$, using the continuity of $\mathcal{T}$ and Lemma 3. Hence $(x_1, 0) = 1_{L^*}$, which is a contradiction to our assumption that $x \neq 1_{L^*}$.

- $\mathcal{T}$ is strong nilpotent.

From Lemma 3 follows $\mathcal{T}(x, y) = 0_{L^*} \iff x \leqslant_{L^*} \mathcal{N}(y)$. So let $x \in D \setminus \{0_{L^*}, 1_{L^*}\}$, then $\mathcal{N}(x) \in D \setminus \{0_{L^*}, 1_{L^*}\}$ (cf. [29]) and $\mathcal{T}(x, \mathcal{N}(x)) = 0_{L^*}$. Hence $\mathcal{T}$ is strong nilpotent.

- $\mathcal{I}_\mathcal{T}(D, D) \subseteq D$.

By Theorem 14, $\mathcal{I}_\mathcal{T}(x, y) = \mathcal{N}_{\mathcal{I}_\mathcal{T}}(\mathcal{T}(x, \mathcal{N}(y))$ for all $x, y \in L^*$. By Theorem 1, there exists an involutive negator N on $[0, 1]$ such that $\mathcal{N}_{\mathcal{I}_\mathcal{T}}(x) = (N(1 - x_2), 1 - N(x_1))$ for all $x \in L^*$. Particularly, if $x \in D$, then $\mathcal{N}_{\mathcal{I}_\mathcal{T}}(x) = (N(x_1), 1 - N(x_1))$. Hence $\mathcal{N}_{\mathcal{I}_\mathcal{T}}(x) \in D$. Since $\mathcal{T}(D, D) \subseteq D$, it follows that $\mathcal{I}_\mathcal{T}(D, D) \subseteq D$. □

Theorem 15. *Let $\mathcal{T}$ be a continuous t-norm on L^* satisfying the residuation principle and $\mathcal{T}(D, D) \subseteq D$. $\mathcal{I}_\mathcal{T}$ is contrapositive if and only if there exists a continuous increasing permutation Φ of L^* such that Φ^{-1} is increasing and such that $\mathcal{T} = \Phi^{-1} \circ \mathcal{T}_W \circ (\Phi \circ \mathrm{pr}_1, \Phi \circ \mathrm{pr}_2)$. If such a Φ exists then $\mathcal{I}_\mathcal{T} = \Phi^{-1} \circ \mathcal{I}_{\mathcal{T}_W} \circ (\Phi \circ \mathrm{pr}_1, \Phi \circ \mathrm{pr}_2)$.*

Proof. Let $\mathcal{T}$ be a continuous t-norm on L^* satisfying the residuation principle and $\mathcal{T}(D, D) \subseteq D$. If $\mathcal{I}_\mathcal{T}$ is contrapositive, then, by Theorem 14, $\mathcal{T}$ satisfies the exchange principle. By Lemma 3 and Theorem 3, $\mathcal{T} = \Phi^{-1} \circ \mathcal{T}_W \circ (\Phi \circ \mathrm{pr}_1, \Phi \circ \mathrm{pr}_2)$.

We obtain, for $\gamma \in L^*$,

$$\begin{aligned}
\mathcal{T}(x, \gamma) \leqslant_{L^*} y &\iff \Phi^{-1}(\mathcal{T}_W(\Phi(x), \Phi(\gamma))) \leqslant_{L^*} y \\
&\iff \mathcal{T}_W(\Phi(x), \Phi(\gamma)) \leqslant_{L^*} \Phi(y) \\
&\iff \Phi(\gamma) \leqslant_{L^*} \mathcal{I}_{\mathcal{T}_W}(\Phi(x), \Phi(y)) \\
&\iff \gamma \leqslant_{L^*} \Phi^{-1}(\mathcal{I}_{\mathcal{T}_W}(\Phi(x), \Phi(y)))
\end{aligned}$$

where we used the fact that $\mathcal{T}_W$ satisfies the residuation principle, and that Φ and Φ^{-1} are increasing permutations. We easily obtain that $\mathcal{I}_{\mathcal{T}_W}(x, y) = \sup\{\gamma \in L^* | \gamma \leqslant_{L^*} \Phi^{-1}(\mathcal{I}_{\mathcal{T}_W}(\Phi(x), \Phi(y)))\} = \Phi^{-1}(\mathcal{I}_{\mathcal{T}_W}(\Phi(x), \Phi(y)))$. Hence $\mathcal{I}_\mathcal{T} = \Phi^{-1} \circ \mathcal{I}_{\mathcal{T}_W} \circ (\Phi \circ \mathrm{pr}_1, \Phi \circ \mathrm{pr}_2)$.

On the other hand, suppose $\mathcal{T} = \Phi^{-1} \circ \mathcal{T}_W \circ (\Phi \circ \mathrm{pr}_1, \Phi \circ \mathrm{pr}_2)$. $\mathcal{I}_\mathcal{T}$ can only be contrapositive w.r.t. its induced negator $\mathcal{N} = \mathcal{N}_{\mathcal{I}_\mathcal{T}}$, defined by $\mathcal{N}_{\mathcal{I}_\mathcal{T}}(x) = \mathcal{I}_\mathcal{T}(x, 0_{L^*})$ for all $x \in L^*$. Then

$$\begin{aligned}\mathcal{N}(x) &= \Phi^{-1}(\mathcal{I}_{\mathcal{T}_W}(\Phi(x), \Phi(0_{L^*}))) \\ &= \Phi^{-1}(\mathcal{I}_{\mathcal{T}_W}(\Phi(x), 0_{L^*})) \\ &= \Phi^{-1}(\mathcal{N}_s(\Phi(x)))\end{aligned}$$

where we used $\mathcal{N}_{\mathcal{I}_{\mathcal{T}_W}} = \mathcal{N}_s$. Hence we obtain

$$\begin{aligned}\mathcal{I}_{\mathcal{T}}(\mathcal{N}(y), \mathcal{N}(x)) &= \Phi^{-1}(\mathcal{I}_{\mathcal{T}_W}(\Phi(\mathcal{N}(y)), \Phi(\mathcal{N}(x)))) \\ &= \Phi^{-1}(\mathcal{I}_{\mathcal{T}_W}(\mathcal{N}_s(\Phi(y)), \mathcal{N}_s(\Phi(x)))) \\ &= \Phi^{-1}(\mathcal{I}_{\mathcal{T}_W}(\Phi(x), \Phi(y))) = \mathcal{I}_{\mathcal{T}}(x, y)\end{aligned}$$

using the fact that $\mathcal{I}_{\mathcal{T}_W}$ is contrapositive w.r.t. $\mathcal{N}_s$. □

Theorem 16. *If $\mathcal{I}$ is a Łukasiewicz implicator on L^* such that $\mathcal{I}(D, D) \subseteq D$, then there exists a continuous increasing permutation Φ of L^* such that Φ^{-1} is increasing and*

$$\mathcal{I}(x, y) = \Phi^{-1} \circ \mathcal{I}_{\mathcal{T}_W} \circ (\Phi \circ \mathrm{pr}_1, \Phi \circ \mathrm{pr}_2)$$

Proof. Since any model implicator on L^* is an S-implicator, we know that $\mathcal{I} = \mathcal{I}_{\mathcal{S},\mathcal{N}}$ for some t-conorm $\mathcal{S}$ on L^* and the involutive negator $\mathcal{N} = \mathcal{N}_{\mathcal{I}}$.

Since $\mathcal{I}$ satisfies (A.5), $x \leqslant_{L^*} y \iff \mathcal{S}(\mathcal{N}(x), y) = 1_{L^*}$, or equivalently $\mathcal{N}(x) \leqslant_{L^*} y \iff \mathcal{S}(x, y) = 1_{L^*}$. Since $\mathcal{S}$ is associative, it holds that $\mathcal{S}(x, \mathcal{S}(y, z)) = 1_{L^*} \iff \mathcal{S}(y, \mathcal{S}(x, z)) = 1_{L^*}$, hence $\mathcal{N}(x) \leqslant_{L^*} \mathcal{S}(y, z) \iff \mathcal{N}(y) \leqslant_{L^*} \mathcal{S}(x, z)$. By changing the variable names as $z \to x$, $y \to y$ and $x \to \mathcal{N}(z)$, we obtain

$$z \leqslant_{L^*} \mathcal{S}(x, y) \iff \mathcal{N}(y) \leqslant_{L^*} \mathcal{S}(x, \mathcal{N}(z)) \tag{6}$$

Let $\mathcal{T}$ be the dual t-norm of $\mathcal{S}$ w.r.t. $\mathcal{N}$, i.e. $\mathcal{T}(x, y) = \mathcal{N}(\mathcal{S}(\mathcal{N}(x), \mathcal{N}(y)))$ for all $x, y \in L^*$. Then (6) is equivalent to $z \leqslant_{L^*} \mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y))) \iff \mathcal{N}(y) \leqslant_{L^*} \mathcal{N}(\mathcal{T}(\mathcal{N}(x), z))$. Since $\mathcal{N}$ is involutive and decreasing, this yields $\mathcal{N}(z) \geqslant_{L^*} \mathcal{T}(\mathcal{N}(x), \mathcal{N}(y)) \iff y \geqslant_{L^*} \mathcal{T}(\mathcal{N}(x), z)$. By substituting the variable names as $x \to \mathcal{N}(x)$, $y \to \mathcal{N}(y)$ and $z \to \mathcal{N}(z)$, we obtain

$$\mathcal{T}(x, y) \leqslant_{L^*} z \iff \mathcal{T}(x, \mathcal{N}(z)) \leqslant_{L^*} \mathcal{N}(y) \tag{7}$$

From Theorem 14 follows that the R-implicator $\mathcal{I}_{\mathcal{T}}$ generated by $\mathcal{T}$ satisfies $\mathcal{I}_{\mathcal{T}}(x, y) = \mathcal{N}(\mathcal{T}(x, \mathcal{N}(y)))$ for all $x, y \in L^*$. Hence $\mathcal{I}_{\mathcal{T}}(x, y) = \mathcal{N}(\mathcal{T}(x, \mathcal{N}(y))) = \mathcal{N}(\mathcal{T}(\mathcal{N}(\mathcal{N}(x)), \mathcal{N}(y))) = \mathcal{S}(\mathcal{N}(x), y) = \mathcal{I}_{\mathcal{S},\mathcal{N}}(x, y) = \mathcal{I}(x, y)$ for all $x, y \in L^*$. Since $\mathcal{I}$ and $\mathcal{N}$ are continuous, so are $\mathcal{S}$ and $\mathcal{T}$.

Since $\mathcal{I}(D, D) \subseteq D$ and $\mathcal{N}(D) \subseteq D$ for all $x, y \in D$ it holds that $\mathcal{S}(x, y) = \mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathcal{N}(x), y) = \mathcal{I}(\mathcal{N}(x), y) \in D$. So we obtain that for all $x, y \in D$ it holds that $\mathcal{T}(x, y) = \mathcal{N}(\mathcal{S}(\mathcal{N}(x), \mathcal{N}(y))) \in D$, i.e. $\mathcal{T}(D, D) \subseteq D$. From Theorems 14 and 15 follows that there exists a continous increasing permutation

Φ of L^* such that Φ^{-1} is increasing and such that $\mathscr{I} = \Phi^{-1} \circ \mathscr{I}_{\mathscr{T}_W} \circ (\Phi \circ \mathrm{pr}_1, \Phi \circ \mathrm{pr}_2)$. $\square$

In [29] it is shown that if Φ is a continuous, increasing permutation of L^* such that Φ^{-1} is also increasing, then there exists a continuous, increasing permutation φ of $[0,1]$ such that $\Phi(x) = (\varphi(x_1), 1 - \varphi(1 - x_2))$ for all $x \in L^*$. It follows that if $\mathscr{I}$ is a Łukasiewicz implicator on L^* such that $\mathscr{I}(D,D) \subseteq D$, then there exists an increasing permutation φ of L^* such that, for all $x, y \in L^*$,

$$\mathscr{I}(x,y) = (\varphi^{-1}\min(1, 1 + \varphi(y_1) - \varphi(x_1), 1 + \varphi(1 - y_2) - \varphi(1 - x_2)), 1 - \varphi^{-1}\min(1, 1 - \varphi(x_1) + \varphi(1 - y_2)))$$

Theorem 17. *Let $\mathscr{I}$ be the implicator on L^* defined by, for all $x, y \in L^*$,*

$$\mathscr{I}(x,y) = (\varphi^{-1}\min(1, 1 + \varphi(y_1) - \varphi(x_1), 1 + \varphi(1 - y_2) - \varphi(1 - x_2)), 1 - \varphi^{-1}\min(1, 1 - \varphi(x_1) + \varphi(1 - y_2)))$$

Then $\mathscr{I}$ is a Łukasiewicz implicator.

Proof. We have $\mathscr{I}(0_{L^*}, y) = (\varphi^{-1}\min(1, 1 + \varphi(y_1), 1 + \varphi(1 - y_2)), 1 - \varphi^{-1} \times \min(1, 1 + \varphi(1 - y_2))) = 1_{L^*}$, $\mathscr{I}(x, 1_{L^*}) = (\varphi^{-1}\min(1, 1 + 1 - \varphi(x_1), 1 + 1 - \varphi(1 - x_2)), 1 - \varphi^{-1}\min(1, 1 - \varphi(x_1) + 1)) = 1_{L^*}$ for all $x, y \in L^*$. Also, $\mathscr{I}(1_{L^*}, 0_{L^*}) = 0_{L^*}$. Clearly, $\mathscr{I}$ is decreasing in its first and increasing in its second component, since φ and φ^{-1} are increasing. Hence $\mathscr{I}$ is an implicator on L^*.

Now $\mathscr{I}(1_{L^*}, y) = (\varphi^{-1}\min(1, 1 + \varphi(y_1) - 1, 1 + \varphi(1 - y_2) - 1), 1 - \varphi^{-1} \times \min(1, 1 - 1 + \varphi(1 - y_2))) = (y_1, y_2) = y$. Hence (A.2) is verified.

For all $x \in L^*$ we have $\mathscr{N}(x) = \mathscr{I}(x, 0_{L^*}) = (\varphi^{-1}\min(1 - \varphi(x_1), 1 - \varphi(1 - x_2)), 1 - \varphi^{-1}(1 - \varphi(x_1))) = (\varphi^{-1}(1 - \varphi(1 - x_2)), 1 - \varphi^{-1}(1 - \varphi(x_1)))$, using the fact that $x_1 \leqslant 1 - x_2$ and φ is increasing. We obtain $\mathscr{I}(\mathscr{N}(y), \mathscr{N}(x)) = (\varphi^{-1}\min(1, 1 + 1 - \varphi(1 - x_2) - 1 + \varphi(1 - y_2), 1 + 1 - \varphi(x_1) - 1 + \varphi(y_1)), 1 - \varphi^{-1} \times \min(1, 1 - 1 + \varphi(1 - y_2) + 1 - \varphi(x_1))) = \mathscr{I}(x,y)$. Hence (A.3) holds.

We have $\mathscr{I}(x, \mathscr{I}(y,z)) = (\varphi^{-1}\min(1, 1 + \varphi(\varphi^{-1}\min(1, 1 + \varphi(z_1) - \varphi(y_1), 1 + \varphi(1 - z_2) - \varphi(1 - y_2))) - \varphi(x_1), 1 + \varphi(1 - 1 + \varphi^{-1}\min(1, 1 - \varphi(y_1) + \varphi(1 - z_2))) - \varphi(1 - x_2)), 1 - \varphi^{-1}\min(1, 1 - \varphi(x_1) + \varphi(1 - 1 + \varphi^{-1}\min(1, 1 - \varphi(y_1) + \varphi(1 - z_2))))) = (\varphi^{-1}\min(1, 1 + 1 + \varphi(z_1) - \varphi(y_1) - \varphi(x_1), 1 + 1 + \varphi(1 - z_2) - \varphi(1 - y_2) - \varphi(x_1), 1 + 1 - \varphi(y_1) + \varphi(1 - z_2) - \varphi(1 - x_2)), 1 - \varphi^{-1}\min(1, 1 - \varphi(x_1) + 1 - \varphi(y_1) + \varphi(1 - z_2)))$, which is symmetrical in x and y and thus equal to $\mathscr{I}(y, \mathscr{I}(x,z))$. So (A.4) holds.

Furthermore, $\mathscr{I}(x,y) = (\varphi^{-1}\min(1, 1 + \varphi(y_1) - \varphi(x_1), 1 + \varphi(1 - y_2) - \varphi(1 - x_2)), 1 - \varphi^{-1}\min(1, 1 - \varphi(x_1) + \varphi(1 - y_2))) = 1_{L^*} \Longleftrightarrow \min(1, 1 + \varphi(y_1) - \varphi(x_1), 1 + \varphi(1 - y_2) - \varphi(1 - x_2)) = 1$ and $\min(1, 1 - \varphi(x_1) + \varphi(1 - y_2)) = 1 \Longleftrightarrow \varphi(y_1) \geqslant \varphi(x_1)$ and $\varphi(1 - y_2) \geqslant \varphi(1 - x_2)$ and $\varphi(1 - y_2) \geqslant \varphi(x_1) \Longleftrightarrow y_1 \geqslant x_1$

and $1 - y_2 \geqslant 1 - x_2$ and $1 - y_2 \geqslant x_1 \iff y_1 \geqslant x_1$ and $y_2 \leqslant x_2 \iff x \leqslant_{L^*} y$ using the fact that φ is an increasing permutation. Hence $\mathscr{I}$ satisfies (A.5).

Since φ is a continuous increasing permutation of $[0, 1]$, φ^{-1} is also continuous. It follows easily that $\mathscr{I}$ is continuous and hence satisfies (A.6). □

The following theorem summarizes the results of this subsection:

Theorem 18. *An $(L^*)^2 \to L^*$ mapping $\mathscr{I}$ is a Łukasiewicz implicator satisfying $\mathscr{I}(D, D) \subseteq D$ if and only if, for all $x, y \in L^*$*

$$\mathscr{I}(x,y) = (\varphi^{-1} \min(1, 1 + \varphi(y_1) - \varphi(x_1), 1 + \varphi(1 - y_2) - \varphi(1 - x_2)),$$
$$1 - \varphi^{-1} \min(1, 1 - \varphi(x_1) + \varphi(1 - y_2)))$$

Open problem. Does there exist a Łukasiewicz implicator such that $\mathscr{I}(D, D) \nsubseteq$ D?

Note. We wish to stress that the axioms imposed by Smets and Magrez are by no means the only interesting ones for implicators. For instance, Gargov and Atanassov [3] enforced the following distributivity condition on $\mathscr{I}$:

$$(\forall x, y, z \in L^*)(\mathscr{I}(x, \mathscr{I}(y, z)) = \mathscr{I}(\mathscr{I}(x, y), \mathscr{I}(x, z))) \tag{8}$$

Łukasiewicz implicators do not satisfy it, while $\mathscr{I}_{\text{ag}}$ from Example 4 does. So if (8) is needed, $\mathscr{I}_{\text{ag}}$ is obviously a better option, yet one should realize that it is not contrapositive.

5. Links with residuated lattices and MV-algebras

It is well-known that classical logic can be described by a boolean algebra. In order to have more than two values for the evaluation of formulas, there have been many attempts to generalize this traditional structure. In this context, residuated lattices and MV-algebras take a particularly distinguished role (see e.g. [38,52,53,62]). The aim of this paragraph is to embed the classification results established above into these general algebraic frameworks for logical calculi.

Definition 12 (*Residuated lattice*). An algebraic structure $\mathscr{L} = (L, \wedge, \vee, \otimes, \to, \mathbf{0}, \mathbf{1})$ is called a residuated lattice provided

- $(L, \wedge, \vee)$ is a bounded lattice with ordering $\leqslant_L$ and $\mathbf{0}$ and $\mathbf{1}$ as its smallest and greatest element, respectively;
- $\otimes$ is a monotonous, commutative, associative $L^2 \to L$ mapping;
- $\to$ is another $L^2 \to L$ mapping such that for all $x, y, z \in L$ holds:

$$x \otimes y \leqslant_L z \iff x \leqslant_L y \to z$$

From this definition and our previous discussion, it is obvious that $(L^*, \mathrm{Min}, \mathrm{Max}, \mathscr{T}, \mathscr{I}_{\mathscr{T}}, 0_{L^*}, 1_{L^*})$ is a residuated lattice if and only if $\mathscr{T}$ is a triangular norm on L^* that satisfies the residuation principle. As a noteworthy example, Łukasiewicz implicators $\mathscr{I}$ such that $\mathscr{I}(D, D) \subseteq D$ (which are always R-implicators generated by a t-norm $\mathscr{T}$ satisfying the residuation principle, cf. Theorem 16) can be used to construct instances of residuated lattices. We denote them by $\mathscr{L}_{\mathscr{I}}$.

Höhle introduced the concept of a square root [39] function of a residuated lattice. Concretely, given the residuated lattice $\mathscr{L} = (L, \wedge, \vee, \otimes, \rightarrow, \mathbf{0}, \mathbf{1})$, an $L \rightarrow L$ mapping $\sqrt{\ }$ is called square root function of $\mathscr{L}$ if for every $x, y \in L$ the following conditions hold:

$$S1 \quad \sqrt{x} \otimes \sqrt{x} = x$$

$$S2 \quad y \otimes y \leqslant_L x \Rightarrow y \leqslant_L \sqrt{x}$$

For the residuated lattice $([0,1], \min, \max, T_W, I_a, 0, 1)$ a square root function exists [52]. It is therefore worthwhile to examine whether this operation is still definable for its extension, the residuated lattice $\mathscr{L}_{\mathscr{I}_{\mathscr{T}_W}}$. The answer turns out to be negative.

Example 12. Assume the square root function of $\mathscr{L}_{\mathscr{I}_{\mathscr{T}_W}}$ exists. Then for all $x \in L^*$ there exists an $x' \in L^*$ such that

$$\mathscr{T}_W(x', x') = (\max(0, 2x'_1 - 1), \min(1, x'_2 + 1 - x'_1)) = x$$

Let $x = (0.1, 0.1)$, then $\mathscr{T}_W(x', x') = x$ if and only if $2x'_1 - 1 = 0.1$ and $x'_2 + 1 - x'_1 = 0.1$, i.e. if $x'_1 = 0.55$ and $x'_2 = 0.1 - 1 + x'_1 = -0.35$, so $x' \notin L^*$. We conclude that no square root function for $\mathscr{L}_{\mathscr{I}_{\mathscr{T}_W}}$ exists.

In [10], Chang defined a stronger version of a residuated lattice called an MV-algebra. We will not reproduce Chang's original, lengthy definition here, but instead define the notion by a characterization in terms of residuated lattices [52].

Definition 13 (*MV-algebra*). An MV-algebra is a residuated lattice $(L, \wedge, \vee, \otimes, \rightarrow, \mathbf{0}, \mathbf{1})$ such that the following condition is fulfilled, for all $x, y \in L$:

$$(x \rightarrow y) \rightarrow y = x \vee y$$

Again the question arises whether the extension $\mathscr{L}_{\mathscr{I}_{\mathscr{T}_W}}$ of ([0, 1],min, max,T_W,I_a,0,1) inherits the property of being an MV-algebra. A simple counterexample shows that it does not. Indeed, let $x = \left(\frac{1}{9}, \frac{1}{5}\right)$ and $y = \left(\frac{2}{3}, \frac{1}{4}\right)$. Then $\mathscr{I}_{\mathscr{T}_W}(\mathscr{I}_{\mathscr{T}_W}(x, y), y) = \left(\frac{43}{60}, \frac{1}{5}\right)$, while $\mathrm{Max}(x, y) = \left(\frac{2}{3}, \frac{1}{5}\right)$. So, whereas the Smets–Magrez axioms can be maintained under the extension to L^*, the property of being an MV-algebra is lost. A stronger result can be proven using Höhle's claim [40] that an infinite, locally finite, [6] complete [7] MV-algebra is necessarily isomorphic with ([0, 1],min,max,T_W,$\mathscr{I}_a$,0,1):

Theorem 19. *If* $\mathscr{L} = (L^*, \text{Min}, \text{Max}, \mathscr{T}, \mathscr{I}_{\mathscr{T}}, 0_{L^*}, 1_{L^*})$ *is locally finite, and* $\mathscr{T}$ *is continuous, then* $\mathscr{L}$ *is no MV-algebra.*

Proof. If $\mathscr{L}$ is an MV-algebra, then $\mathscr{I}_{\mathscr{T}}$ must be contrapositive [52]. From Theorem 14, we derive that $\mathscr{T}$ satisfies the exchange principle, and thus by Lemma 7 and the continuity of $\mathscr{T}$, $\mathscr{T}$ is archimedean. Since no isomorphism exists between [0, 1] and L^*, the proof is complete provided $\mathscr{L}$ is locally finite. □

6. Intuitionistic fuzzy and interval-valued implicators: applications, opportunities, challenges

During the past two decades, fuzzy implicators have played an increasingly prominent role within the research focused on fuzzy sets and fuzzy logic. Fuzzy implicators have been shown useful in rule-based (expert) systems, for defining inclusion measures and measures of guaranteed possibility and necessity, for characterizing fuzzy rough sets and linguistic modifiers, for pattern classification, aggregation, preference modelling and decision making, fuzzy logic programming and many other fields.

Our goal in these pages is not to go into the details of generalizing all or even any of these application domains; let it suffice to mention that a lot of work in that direction has already been done in both the interval computations (see e.g. [11,12,37,43,44,47,51]) and intuitionistic fuzzy set theory communities (see e.g. [4,7,8,20,21,50,56]). Rather we aim to provide some basic insight of (a) the usefulness and (b) the flexibility of calculation obtained by our lattice-valued approach to implicators, and to outline some challenges and opportunities that future research in this area could face. As a running example throughout this section, we take the well-known generalized modus ponens (GMP) inference rule [12,24,17], which reads:

IF X is A, THEN	Y is B	(1)
X is A'		(2)
	Y is B'	(3)

[6] An MV-algebra $(L, \wedge, \vee, \otimes, \to, \mathbf{0}, \mathbf{1})$ is called locally finite if to every $x \in L \setminus \{\mathbf{1}\}$, there exists an $n \in \mathbb{N}$ such that

$$x^n = \underbrace{x \otimes x \otimes \cdots \otimes x}_{n\text{times}} = \mathbf{0}$$

[7] An MV-algebra with evaluation set L is called complete if L is complete.

X and Y are assumed to be variables in the respective universes U and V. A and A' are mappings from U to L^* (elements of $\mathscr{F}_{L^*}(U)$), while B and B' are mappings from V to L^* (elements of $\mathscr{F}_{L^*}(V)$), and all of them can be interpreted as interval-valued or intuitionistic fuzzy sets. Generally, they are assumed to be *normalized*: an L^*-fuzzy set A in U is called normalized if there exists at least one $u \in U$ such that $A(u) = 1_{L^*}$. The statements (1) and (2) above the line are called the if-then rule and the observation on X, respectively, while the statement (3) below the line is called the inference on Y. The above scheme does not state what the fuzzy restriction B' should be when A, A' and B are given. From all the possible alternatives, we consider here only the single best-known one, i.e. the implementation by the compositional rule of inference (CRI). This rule is a rigorous tool from relational calculus (see e.g. [15]) that joins the if–then rule and the observation and projects the result onto the universe of Y. Using the notation of [17], if $\mathscr{T}$ is a t-norm on L^*, $\mathscr{I}$ an implicator on L^* and $\mathscr{I}(A,B)$ is the mapping from $U \times V$ to L^* defined by, for all $(u,v) \in U \times V$

$$\mathscr{I}(A,B)(u,v) = \mathscr{I}(A(u),B(v))$$

then we define the operator (actually a mapping from $\mathscr{F}_{L^*}(X)$ to $\mathscr{F}_{L^*}(Y)$) $\mathrm{cri}^{\mathscr{T}}_{\mathscr{I}(A,B)}$ for all $v \in V$ by

$$\mathrm{cri}^{\mathscr{T}}_{\mathscr{I}(A,B)}(A')(v) = \sup_{u\in U} \mathscr{T}(A'(u), \mathscr{I}(A(u),B(v)))$$

Note how smoothly the extension of the GMP and the CRI [8] is obtained. Again, this is due to the nature of L^*-fuzzy sets (and, more generally, arbitrary L-fuzzy sets) that allow all of the order-theoretic notions such as conjunction, composition, ...to be straightforwardly defined on them.

As a first simple consistency test for the procedure defined above, we investigate under which conditions it extends the classical modus ponens, that is: if $A' = A$, then $B' = B$.

Theorem 20. *If $\mathscr{T}$ is a t-norm on L^* satisfying the residuation principle, then $\mathrm{cri}^{\mathscr{T}}_{\mathscr{I}_{\mathscr{T}}(A,B)}(A) = B$ for all normalized L^*-fuzzy sets A and B.*

Proof. In [29] the following claim was proven: a t-norm $\mathscr{T}$ on L^* satisfies the residuation principle if and only if

$$\sup_{z\in Z} \mathscr{T}(x,z) = \mathscr{T}(x, \sup_{z\in Z} z) \tag{9}$$

for any $x \in L^*$ and any subset Z of L^*. This allows for the following deduction, for $u \in U$ and $v \in V$:

[8] In general, $\mathscr{I}(A,B)$ can be replaced by any $U \times V \to L^*$ mapping R.

$$\begin{aligned}\mathscr{T}(A(u),\mathscr{I}_{\mathscr{T}}(A(u),B(v))) &= \mathscr{T}(A(u),\sup\{\gamma\in L^*|\mathscr{T}(A(u),\gamma)\leqslant_{L^*}B(v)\})\\ &= \sup\{\mathscr{T}(A(u),\gamma)|\gamma\in L^* \text{ and } \mathscr{T}(A(u),\gamma)\\ &\leqslant_{L^*}B(v)\}\leqslant_{L^*}B(v)\end{aligned}$$

On the other hand, since A is normalized, there exists $u^*\in U$ such that $A(u^*)=1_{L^*}$, so $\mathscr{T}(A(u^*),\mathscr{I}_{\mathscr{T}}(A(u^*),B(v)))=\mathscr{T}(1_{L^*},\mathscr{I}_{\mathscr{T}}(1_{L^*},B(v))=B(v)$ since $\mathscr{T}$ is a t-norm and $\mathscr{I}_{\mathscr{T}}$ is a border implicator. Hence, for all $v\in V$,

$$\mathrm{cri}^{\mathscr{T}}_{\mathscr{I}_{\mathscr{T}}(A,B)}(A)(v)=B(v) \qquad \square$$

Note. It is important to mark the difference with the analogous situation in fuzzy set theory, where left-continuity of a t-norm T on $[0,1]$ sufficed to extend the modus ponens if the associated R-implicator was used. In [29], however, a continuous t-norm on L^* was constructed that does not satisfy equality (9).

In the next two subsections, we will come back to the GMP example to illustrate the incorporation of uncertainty by means of IFSs and IVFSs into knowledge-manipulating processes.

6.1. Propagation of uncertainty in L^*-fuzzy sets

It has been mentioned in the introduction that fuzzy sets are unable to deal adequately with uncertainty. In this light, a degree of non-determinacy π was introduced. In this paragraph, we will treat π as a mapping from L^* to $[0,1]$ defined by, for $x\in L^*$, $\pi(x)=1-x_1-x_2$. For values x in D, obviously $\pi(x)=0$.

The mapping π has the interesting feature that it allows us to focus exclusively on uncertainty: controlling the propagation of uncertainty in the GMP example will be tantamount to controlling the values $\pi(x)$ for the membership degrees x to the result L^*-fuzzy set B'. In this sense, we lend a willing ear to the constraints imposed on the implicator $\mathscr{I}$ by Bustince et al. in [9], for $x,y\in L^*$:

$B.1 \quad \pi(\mathscr{I}(x,y))\leqslant \max(1-x_1,1-y_1)$

$B.2 \quad x=y\Rightarrow \pi(\mathscr{I}(x,y))=\pi(x)$

$B.3 \quad \pi(x)=\pi(y)\Rightarrow \pi(\mathscr{I}(x,y))=\pi(x)$

The first constraint is aimed at establishing upper bounds for the uncertainty caused by an application of an implicator. The following easy deduction shows that criterion (B.1) is satisfied by all Łukasiewicz implicators, and more generally by all S-implicators $\mathscr{I}_{\mathscr{S},\mathscr{N}}$ on L^* generated by arbitrary involutive $\mathscr{N}$ and $\mathscr{S}$, *by a very comfortable margin*:

$$\begin{aligned}\pi(\mathscr{I}(x,y)) &= 1 - \mathrm{pr}_1(\mathscr{S}(\mathscr{N}(x),y)) - \mathrm{pr}_2(\mathscr{S}(\mathscr{N}(x),y))\\ &\leqslant 1 - \mathrm{pr}_1(\mathscr{S}(\mathscr{N}(x),y)) = 1 - \mathrm{pr}_1(\mathscr{S}((N(1-x_2),1-N(x_1)),y))\\ &\leqslant 1 - \max(N(1-x_2),y_1) = \min(1-N(1-x_2),1-y_1)\\ &\leqslant \max(1-x_1,1-y_1)\end{aligned}$$

where we used the representation Theorem 1 for $\mathscr{N}$ by the fuzzy negator N. Obviously, the one but last line of the above deduction offers a tighter criterion that we propose to replace the right hand side of (B.1) with; it will be hard to prove compliance with stronger restrictions since lower bounds on $\mathscr{I}$'s second projection are difficult to obtain in general.

The remaining conditions address another issue, namely the conservation of uncertainty through the application of an implicator on L^*. Despite their fairness from an intuitive perspective, they conflict with the requirement imposed by logic that $\mathscr{I}(x,x) = 1_{L^*}$ for $x \in L^*$, an immediate consequence of axiom (A.5); Łukasiewicz implicators therefore never satisfy (B.2) nor (B.3); the implicator from Example 2 on the other hand does.

The above observation appears to point out an intuitive anomaly, namely that *uncertainty can disappear entirely through the application of an implicator on* L^*. Although this important debate on the apparent clash between logical and cognitive assumptions needs to be pursued further and in full depth, our feeling is that a decision to comply with (B.2) and/or (B.3) should depend primarily on the application at hand, and so the conditions are not absolute. For instance, in determining the degree of inclusion of fuzzy sets into one another, sometimes the formula [26]

$$\mathrm{Inc}(A,B) = \inf_{u\in U} I(A(u),B(u))$$

is used, where A and B are fuzzy sets in U and I is an implicator on $[0,1]$. Suppose we replace I by an implicator $\mathscr{I}$ on L^* and generalize A, B to arbitrary L^*-fuzzy sets. Naturally, $\mathrm{Inc}(A,B)$ should be equal to 1 if and only if $A \subseteq B$, that is $(\forall u \in U)(A(u) \leqslant_{L^*} B(u))$. Hence $\mathscr{I}$ should satisfy (A.5). This material is studied in detail in [25].

In the GMP example, there do not seem to be any arguments in favour of (B.2) and (B.3) either. A more relevant criterion is the following:

$$(\forall v \in V)(\pi(\mathrm{cri}^{\mathscr{T}}_{\mathscr{I}(A,B)}(A')(v)) \geqslant \pi(B(v)))$$

In other words, unreliable observations do not give way to strictly more reliable conclusions, or equivalently uncertainty does not vanish. It can be equivalently stated as:

$$\mathrm{pr}_1(\mathrm{cri}^{\mathscr{T}}_{\mathscr{I}(A,B)}(A')(v)) + \mathrm{pr}_2(\mathrm{cri}^{\mathscr{T}}_{\mathscr{I}(A,B)}(A')(v)) \geqslant \mathrm{pr}_1(B(v)) + \mathrm{pr}_2(B(v))$$

Unfortunately, with the current machinery of Łukasiewicz implicators, the latter inequality seems very difficult to realize. The main problem is that there is no direct correlation between π and the ordering in L^*, so that even when $x \leqslant_{L^*} y$, all three options: $\pi(x) = \pi(y)$, $\pi(x) > \pi(y)$, $\pi(x) < \pi(y)$ are still possible. We postpone the algebraic investigation of this property to a future paper, remarking that the criterion is trivially met if there is no uncertainty in the if–then rule ($\mathscr{I}(A(u), B(v)) \in D$ for all $(u, v) \in U \times V$). This section identified one index of uncertainty associated with L^*-fuzzy sets; in the next subsection we fit these ideas into a larger framework focussed on dealing with imprecise knowledge.

6.2. A view of uncertainty inspired by intuitionistic fuzzy possibility theory

In [18,19,22], an intuitionistic fuzzy extension of possibility theory (in the sense of Zadeh [68]) was proposed. We briefly recall the main ideas. [9]

The central notion in possibility theory is that of a so-called *elastic restriction* that allows us to discriminate between the more or less plausible values for a variable X in a universe U. In this sense, it reflects our *uncertainty about the true value of* X. This elastic restriction is modelled by a mapping π_X from U to a set L, whose values represent degrees of possibility, so that $\pi_X(u) = l$ means that it is possible to degree $l \in L$ that X takes the value $u \in U$. Yet, typically a mix of positive and negative evidence contributes to our knowledge about X; positive evidence here means that we get information that particular values are to a given extent *possible* for X, while negative evidence includes those statements that tell us something about the *necessity* that X cannot in fact take a particular value. [10] It therefore appears counterintuitive to let this kind of information be represented by a single degree (of possibility) for every element in the universe, thereby enforcing implicit duality of the degree of necessity. Indeed, in traditional possibility theory, where $L = [0, 1]$, two measures of possibility and necessity of a crisp set A in U are defined:

$$\Pi_X(A) = \sup_{u \in A} \pi_X(u)$$

$$N_X(A) = \inf_{u \notin A} 1 - \pi_X(u)$$

Obviously, $N_X(A) = 1 - \Pi_X(\text{co}(A))$, where $\text{co}(A)$ represents the complement of A. It makes more sense to have two separate distributions that define the degree

[9] The mentioned references considered also an interval-valued extension of possibility theory, but we do not consider it here.

[10] Note that this two-sided view bears a likeness to a widely used practise in artificial intelligence (e.g. in learning processes and in fuzzy rough sets), namely to approximate a concept by giving positive and negative specimen for it.

of possibility that $X = u$ and the degree of necessity that $X \neq u$, respectively. In [22] we suggested to use the membership and non-membership functions of an intuitionistic fuzzy set for that purpose, such that formally $L = L^*$. The resulting distribution π_X was called an intuitionistic fuzzy possibility distribution. The altered measures of possibility and necessity now read:

$$\Pi_X(A) = \sup_{u \in A} \mathrm{pr}_1(\pi_X(u))$$

$$N_X(A) = \inf_{u \notin A} \mathrm{pr}_2(\pi_X(u))$$

They satisfy a weakened duality, i.e. $N_X(A) \leqslant 1 - \Pi_X(\mathrm{co}(A))$, embodying the cognitive constraint that our belief (necessity) that $X \neq u$ cannot surpass one minus the possibility that $X = u$. $1 - \Pi_X(A) - N_X(A)$ can be used to model disbelief in (unreliability of) the observer that provided the information.

Let us consider this interpretation in the framework of our GMP example. A, A', B and B' will all represent possibility distributions on their associated variables. Let us assume at this moment that there is no uncertainty present in the if–then rule, so $A(u) \in D$ and $B(v) \in D$ for all $u \in U$ and $v \in V$. An interesting situation to study is one in which we completely discredit the observer, that is $\mathrm{pr}_2(A'(u)) = 0$ for all $u \in U$. An obvious constraint to impose, then, is the non-credibility of the result: $\mathrm{pr}_2(B'(v)) = 0$ for all $v \in V$, since we do not want to be forced to make any commitment due to an unreliable observer. Assuming there exists $u^* \in V$ such that $A(u^*) = 0_{L^*}$, we have

$$\begin{aligned}
\mathrm{cri}^{\mathscr{T}}_{\mathscr{I}(A,B)}(A')(v) &= \sup_{u \in U} \mathscr{T}(A'(u), \mathscr{I}(A(u), B(v))) \\
&\geqslant_{L^*} \mathscr{T}(A'(u^*), \mathscr{I}(A(u^*), B(v))) \\
&= \mathscr{T}(A'(u^*), \mathscr{I}(0_{L^*}, B(v))) \\
&= \mathscr{T}(A'(u^*), 1_{L^*}) \\
&= (\mathrm{pr}_1(A'(u^*)), 0)
\end{aligned}$$

Hence, $\mathrm{pr}_2(\mathrm{cri}^{\mathscr{T}}_{\mathscr{I}(A,B)}(A')(v)) = 0$ and the desired result is obtained. This result holds regardless if $\mathscr{T}$ is t-representable or not.

In general, we could ask how the uncertainty associated with the observer should be reflected in the result. It is important to note that the discussion on the propagation of π-values from the previous subsection does not apply here, because A' and B' are (usually) associated with different variables X and Y. It appears that the CRI already takes care of variations in the possibility distribution associated with X; by enforcing its semantics of inferring the most specific distribution on Y consistent with the constraints on X and Y, it puts bounds on the reliability of the result, expressed by $\mathrm{pr}_2(B')$. Experiments will have to reveal whether this belief is justified, and whether the CRI indeed operates in accordance with our expectations on the propagation of more or less reliable information.

6.3. Other challenges and future work

In this short concluding paragraph, we list a few areas in which further work and/or experiment is mandatory, along with appropriate questions to ask.

- Jenei [41] studied continuity w.r.t. to different metrics in the GMP for a particular subclass of fuzzy quantities (fuzzy sets in $\mathbb{R}$). His results favoured the Łukasiewicz implicators on $[0,1]$. Does this observation extend to L^*?
- Atanassov and Gargov [3,5] proposed to replace the classical notion of a tautology with that of an intuitionistic fuzzy tautology (IFT). A formula P is called an IFT if its truth value $x \in L^*$ is such that $x_1 \geqslant x_2$. Not every IFT is necessarily also a classical tautology. For instance, for the implicator $\mathscr{I}_{\mathrm{Max},\mathscr{N}_s}$ in Example 2, the formula $\mathscr{I}(x,x) = 1_{L^*}$ is not a tautology, but is an IFT since $\max(x_2,x_1) \geqslant \min(x_1,x_2))$. IFTs may allow Bustince et al.'s criteria to coexist with a modified version of the Smets–Magrez axioms. This direction has yet to be explored in full depth. Related to this is Kenevan and Neapolitan's [44], and later Entemann's [33] work on a logic with interval truth values along with a proof theory for it. It was shown that this logic is "fuzzy complete", that is: all fuzzy tautologies, i.e. formulas P such that their truth value $x \in L^*$ satisfies $x_1 \geqslant 0.5$, can be proven in the theory. The question then arises whether we could develop something like "intuitionistic fuzzy completeness".
- IVFSs and IFSs can both be considered as stepping stones in a larger context:
  - IVFSs are characterized by membership degrees which are intervals in $[0,1]$. Going one step further, type-2 fuzzy sets emerge when we allow membership degrees themselves to be fuzzy sets in $[0,1]$. They have been receiving a lot of renewed attention lately, amongst others by Mendel [47] and Türkşen [61], as a vehicle particularly suited to implementing the computing with words (CWW) paradigm. It is worthwhile to further investigate the algebraic structure on which type-2 fuzzy sets are defined. In this sense the theorem by Mizumoto and Tanaka [48] that convex and normalized type-2 fuzzy sets give way to a bounded lattice is very important.
  - IFS theory, which is specifically tuned to the concept of positive (membership) and negative (non-membership) constituents, can be generalized by dropping the restriction that $\mu + \nu \leqslant 1$, and by instead drawing (μ, ν) from $[0,1]^2$. This extension was coined quadrivalent (i.e. four-valued) fuzzy logic and was studied e.g. in [13,34].

7. Conclusion

We have constructed a representation theorem for Łukasiewicz implicators on the lattice L^* which serves as the underlying algebraic structure for both intuitionistic fuzzy and interval-valued fuzzy sets. We have related our results to the general theory of residuated lattices and MV-algebras, and explained how to apply them in a practical context to model of different kinds of imprecision.

Acknowledgements

Chris Cornelis would like to thank the Fund for Scientific Research-Flanders for funding the research elaborated on in this paper.

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