On the Representation of Intuitionistic Fuzzy t-Norms and t-Conorms

Glad Deschrijver, Chris Cornelis, and Etienne E. Kerre

***Abstract*—Intuitionistic fuzzy sets form an extension of fuzzy sets: while fuzzy sets give a degree to which an element belongs to a set, intuitionistic fuzzy sets give both a membership degree and a nonmembership degree. The only constraint on those two degrees is that their sum must be smaller than or equal to 1. In fuzzy set theory, an important class of triangular norms and conorms is the class of continuous Archimedean nilpotent triangular norms and conorms. It has been shown that for such t-norms T there exists a permutation φ of [0,1] such that T is the φ-transform of the Łukasiewicz t-norm. In this paper we introduce the notion of intuitionistic fuzzy t-norm and t-conorm, and investigate under which conditions a similar representation theorem can be obtained.**

***Index Terms*—Archimedean property, intuitionistic fuzzy set, intuitionistic fuzzy triangular norm and conorm, nilpotency, representation theorem, φ-transform.**

I. Introduction

FUZZY set theory has been shown to be a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situations by attributing a degree to which a certain object belongs to a set. In real life, however, a person may assume that an object x belongs to a set A to a certain degree, but it is possible that he is not so sure about it. In other words, there may be a hesitation or uncertainty about the membership degree of x in A. In fuzzy set theory, there is no means to incorporate that hesitation in the membership degrees. A possible solution is to use intuitionistic fuzzy sets, defined by Atanassov in 1983 [1]. Intuitionistic fuzzy sets give us the possibility to model hesitation and uncertainty by using an additional degree. An intuitionistic fuzzy set A assigns to each element u of the universe U a membership degree $\mu_A(u)(\in [0,1])$ and a nonmembership degree $\nu_A(u)(\in [0,1])$ such that $\mu_A(u) + \nu_A(u) \leq 1$. For all $u \in U$, the value $\pi_A(u) = 1 - \mu_A(u) - \nu_A(u)$ is called the hesitation degree or the intuitionistic index of u to A.

In fuzzy set theory, the nonmembership degree of an element x of the universe is defined as one minus the membership degree (using the standard negation) and thus it is fixed. In intuitionistic fuzzy set theory, the nonmembership degree is more or less independent: the only condition is that it is smaller than one minus the membership degree. Note that both μ_A and ν_A can be seen as fuzzy sets on X (that are not completely independent, because of the condition that the sum of the two degrees should be less than or equal to 1). In this way, the negation of the nonmembership degree w.r.t. the standard fuzzy negation can be seen as a degree of membership. So, for each element $u \in U$ there exist two degrees that model the membership of u in the intuitionistic fuzzy set A, namely $\mu_A(u)$ and $1 - \nu_A(u)$. The length of the interval $[\mu_A(u), 1 - \nu_A(u)]$, which is given by $\pi_A(u)$, can then be seen as a degree modeling the hesitation between the two membership degrees.

An important notion in fuzzy set theory is that of triangular norms and conorms: t-norms and t-conorms are used to define a generalized intersection and union of fuzzy sets, they are applied in the compositional rule of inference (CRI) to obtain the result of the generalized modus ponens (GMP) (see [2] and [3] for an intuitionistic fuzzy set-based approach of the CRI), they are used to define fuzzy inclusion measures [4]–[6]. Triangular norms and conorms serve as aggregation operators, which can be used, e.g., for querying databases (see [7] for an approach via intuitionistic fuzzy sets), to handle multiple rules in the GMP [8], to compute the resulting degree of confidence in a hypothesis when the separate degrees to which the experts support the hypothesis are given [9],... Using intuitionistic fuzzy t-norms the composition of fuzzy relations was extended to the intuitionistic fuzzy case (see [10]–[12]); these compositions are useful in approximate reasoning, e.g., for medical diagnosis and information retrieval (see, e.g., [13] and [14]).

In fuzzy set theory continuous, Archimedean, nilpotent t-norms play a very important role (see, e.g., [15]); they occur for instance in the theory of Łukasiewicz implicators, i.e., fuzzy implicators that fulfill the entire axiom set of Smets and Magrez [16]. A representation theorem was established for continuous Archimedean nilpotent t-norms: a t-norm T is continuous, Archimedean and nilpotent if and only if there exists a permutation φ of [0,1] such that T is the φ-transform of the Łukasiewicz t-norm T_W, i.e. $T = \varphi^{-1} \circ T_W \circ (\varphi \times \varphi)$, where T_W is defined as $T_W(x,y) = \max(0, x+y-1)$, for all $x, y \in [0,1]$, and where $\times$ denotes the product operation [17]. An analogous result holds for t-conorms. In this paper, we extend the notions of t-norm and t-conorm to the intuitionistic fuzzy case, and we generalize said representation theorems to these intuitionistic fuzzy connectives.

II. Intuitionistic Fuzzy Sets

Intuitionistic fuzzy sets were introduced by Atanassov in 1983 and are defined as follows.

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The authors are with the Department of Mathematics and Computer Science, Fuzziness and Uncertainty Modeling Research Unit, Ghent University, B-9000 Ghent, Belgium (e-mail: glad.deschrijver@UGent.be; chris.cornelis@UGent.ac.be; etienne.kerre@UGent.ac.be).

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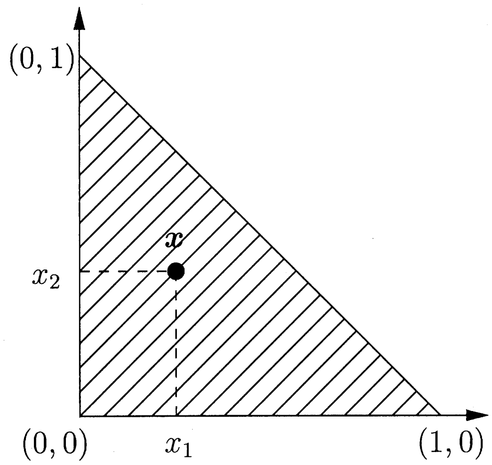

Fig. 1. Graphical representation of the set L^*.

Definition 2.1: [1], [18], [19] An intuitionistic fuzzy set A in a universe U is an object

$$A = \{(u, \mu_A(u), \nu_A(u)) \,|u \in U\}$$

where, for all $u \in U$, $\mu_A(u) \in [0,1]$ and $\nu_A(u) \in [0,1]$ are called the membership degree and the nonmembership degree, respectively, of u in A, and furthermore satisfy $\mu_A(u) + \nu_A(u) \leq 1$.

Deschrijver and Kerre [20] have shown that intuitionistic fuzzy sets can also be seen as L-fuzzy sets in the sense of Goguen [21]. Consider the set L^* and operation $\leq_{L^*}$ defined by

$$L^* = \left\{(x_1, x_2)|(x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\right\}$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2$$
$$\forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then, $(L^*, \leq_{L^*})$ is a complete lattice [20]. For each nonempty $A \subseteq L^*$, we have

$$\begin{aligned}\sup A = (&\sup \{x_1 | x_1 \in [0,1] \text{ and } (\exists x_2 \in [0, 1-x_1]) \\ &((x_1, x_2) \in A)\}, \\ &\inf \{x_2 | x_2 \in [0,1] \text{ and } (\exists x_1 \in [0, 1-x_2]) \\ &((x_1, x_2) \in A\}) \\ \inf A = (&\inf \{x_1 | x_1 \in [0,1] \text{ and } (\exists x_2 \in [0, 1-x_1]) \\ &((x_1, x_2) \in A)\}, \\ &\sup \{x_2 | x_2 \in [0,1] \text{ and } (\exists x_1 \in [0, 1-x_2]) \\ &((x_1, x_2) \in A\}).\end{aligned}$$

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$.

Note that if, for $x = (x_1, x_2)$, $y = (y_1, y_2) \in L^*$, $x_1 < y_1$ and $x_2 < y_2$, then x and y are incomparable w.r.t. $\leq_{L^*}$, denoted as $x||_{L^*}y$.

The shaded area in Fig. 1 is the set of elements $x = (x_1, x_2)$ belonging to L^*. From now on, we will assume that if $x \in L^*$, then x_1 and x_2 denote, respectively, the first and the second component of x, i.e., $x = (x_1, x_2)$.

Equivalently, this lattice can also be defined as an algebraic structure $(L^*, \wedge, \vee)$ where the meet operator $\wedge$ and the join operator $\vee$ are defined as follows, for $x, y \in L^*$:

$$x \wedge y = (\min(x_1, y_1), \max(x_2, y_2))$$
$$x \vee y = (\max(x_1, y_1), \min(x_2, y_2)).$$

Using this lattice, we easily see that with every intuitionistic fuzzy set $A = \{(u, \mu_A(u), \nu_A(u))|u \in U\}$ corresponds an L^*-fuzzy set, i.e., a mapping $A : U \rightarrow L^* : u \mapsto (\mu_A(u), \nu_A(u))$. In the sequel, we will use the same notation for an intuitionistic fuzzy set and its associated L^*-fuzzy set. So, for the intuitionistic fuzzy set A we will also use the notation $A(u) = (\mu_A(u), \nu_A(u))$.

Interpreting intuitionistic fuzzy sets as L^*-fuzzy sets gives way to greater flexibility in calculating with membership and nonmembership degrees, since the pair formed by the two degrees is an element of L^*, and often allows to obtain more compact formulas. Moreover, some operators that are defined in the fuzzy case, such as fuzzy implicators, can be easily extended to the intuitionistic fuzzy case by using the lattice $(L^*, \leq_{L^*})$.

The union of two intuitionistic fuzzy sets A and B in a universe U is defined as

$$A \cup B = \{(u, \max(\mu_A(u), \mu_B(u)), \min(\nu_A(u), \nu_B(u))) \,|u \in U\}.$$

Using L^*, we obtain that $A \cup B(u) = (\max(\mu_A(u), \mu_B(u)), \min(\nu_A(u), \nu_B(u))) = A(u) \vee B(u)$, for all $u \in U$. Similarly the intersection of A and B is defined as

$$A \cap B = \{(u, \min(\mu_A(u), \mu_B(u)), \max(\nu_A(u), \nu_B(u))) \,|u \in U\}$$

or, using L^*, $A \cap B(u) = A(u) \wedge B(u)$, for all $u \in U$.

As an example of how fuzzy operators can be extended to intuitionistic fuzzy operators using the lattice L^*, we give the definition of an intuitionistic fuzzy implicator, since we will also need it in the sequel.

Definition 2.2: [2] An intuitionistic fuzzy implicator is an $(L^*)^2 - L^*$ mapping $\mathcal{I}$ satisfying the following conditions:

$$\mathcal{I}(0_{L^*}, 0_{L^*}) = 1_{L^*} \quad \mathcal{I}(0_{L^*}, 1_{L^*}) = 1_{L^*}$$
$$\mathcal{I}(1_{L^*}, 1_{L^*}) = 1_{L^*} \quad \mathcal{I}(1_{L^*}, 0_{L^*}) = 0_{L^*}$$

$$(\forall y \in L^*) \left(\forall (x, x') \in (L^*)^2\right)$$
$$(x \leq_{L^*} x' \Rightarrow \mathcal{I}(x, y) \geq_{L^*} \mathcal{I}(x', y))$$
$$(\forall x \in L^*) \left(\forall (y, y') \in (L^*)^2\right)$$
$$(y \leq_{L^*} y' \Rightarrow \mathcal{I}(x, y) \leq_{L^*} \mathcal{I}(x, y')).$$

We also define the following set for further usage: $D = \{x | x \in L^* \text{ and } x_1 + x_2 = 1\}$, and the first and second projection mapping pr_1 and pr_2 on L^*, defined as $pr_1(x_1, x_2) = x_1$ and $pr_2(x_1, x_2) = x_2$, for all $(x_1, x_2) \in L^*$.

III. Intuitionistic Fuzzy Negators

Intuitionistic fuzzy negators form an extension of fuzzy negators and are defined as follows.

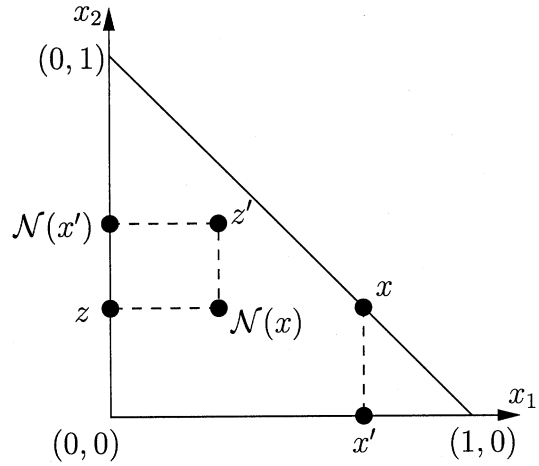

Fig. 2. Proof of Lemma 3.3.

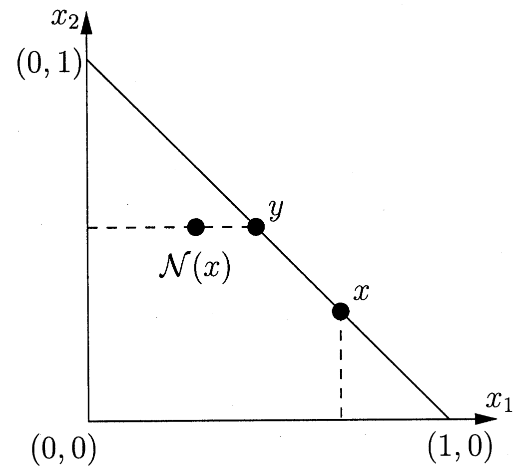

Fig. 3. Proof of Lemma 3.5.

Definition 3.1: An intuitionistic fuzzy negator is any decreasing $L^* - L^*$ mapping $\mathcal{N}$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator.

The mapping $\mathcal{N}_s$ defined by $\mathcal{N}_s(x_1, x_2) = (x_2, x_1)$, for all $(x_1, x_2) \in L^*$, will be called the standard negator.

Now, we establish a representation theorem for involutive intuitionistic fuzzy negators. This representation will be useful to prove some important results in the sequel. To this aim, we introduce some preliminary lemmas.

Lemma 3.1: For any involutive intuitionistic fuzzy negator $\mathcal{N}$ there holds $\mathcal{N}(0,0) = (0,0)$.

Proof: Let $\mathcal{N}$ be an involutive intuitionistic fuzzy negator, and assume that $\mathcal{N}(0,0) \neq (0,0)$. Then, either $\mathcal{N}(0,0) = (a,0)$ where $a \neq 0$, or $\mathcal{N}(0,0) = (0,b)$ where $b \neq 0$, or $\mathcal{N}(0,0) = (a,b)$ where $a \neq 0$ and $b \neq 0$.

Assume $\mathcal{N}(0,0) = (a,0)$ with $a \neq 0$. Then for any $x, x' \in L^*$ such that $x \geq_{L^*} (0,0)$ and $x' \geq_{L^*} (0,0)$, we have either $x \geq_{L^*} x'$ or $x \leq_{L^*} x'$. Let now $y, y' \in L^*$ be such that neither $y \leq_{L^*} y'$ nor $y \geq_{L^*} y'$, but $y \leq_{L^*} (a,0)$ and $y' \leq_{L^*} (a,0)$. Then, since $\mathcal{N}$ is involutive and decreasing, $\mathcal{N}(y) \geq_{L^*} \mathcal{N}(a,0) = (0,0)$ and $\mathcal{N}(y') \geq_{L^*} (0,0)$. But then we have either $\mathcal{N}(y) \geq_{L^*} \mathcal{N}(y')$ or $\mathcal{N}(y) \leq_{L^*} \mathcal{N}(y')$, hence, either $y \leq_{L^*} y'$ or $y \geq_{L^*} y'$, which is a contradiction. In a similar way, we obtain a contradiction in the two other cases. ∎

Corollary 3.2: For any involutive intuitionistic fuzzy negator $\mathcal{N}$ there holds, for all $a \in [0,1]$: $pr_2\mathcal{N}(0,a) = 0$ and $pr_1\mathcal{N}(a,0) = 0$.

Proof: Since $\mathcal{N}$ is decreasing and $(0,a) \leq_{L^*} (0,0)$, we obtain that $\mathcal{N}(0,a) \geq_{L^*} \mathcal{N}(0,0) = (0,0)$, hence, $pr_2\mathcal{N}(0,a) = 0$. Similarly, $pr_1\mathcal{N}(a,0) = 0$. ∎

Lemma 3.3: Let $\mathcal{N}$ be an involutive intuitionistic fuzzy negator. For all $x_1 \in [0,1]$, $pr_2\mathcal{N}(x_1, 1-x_1) = pr_2\mathcal{N}(x_1, 0)$ and $pr_1\mathcal{N}(1-x_1, x_1) = pr_1\mathcal{N}(0, x_1)$.

Proof: Let $x_1 \neq 1$ and $x = (x_1, 1-x_1)$ and $x' = (x_1, 0)$ (if $x_1 = 1$ then the lemma trivially holds). Assume $pr_2\mathcal{N}(x) \neq pr_2\mathcal{N}(x')$, i.e., $pr_2\mathcal{N}(x) < pr_2\mathcal{N}(x')$ (see Fig. 2). For any $y = (x_1, y_2)$, $y' = (x_1, y_2') \in L^*$ we have either $y \leq_{L^*} y'$ or $y \geq_{L^*} y'$. From Corollary 3.2, it follows that $pr_1\mathcal{N}(x') = 0$. Let now $z = (0, pr_2\mathcal{N}(x))$ and $z' = (\min(pr_1\mathcal{N}(x), 1 - pr_2\mathcal{N}(x')), pr_2\mathcal{N}(x'))$. Since $pr_2\mathcal{N}(x) < pr_2\mathcal{N}(x')$, we obtain that $\mathcal{N}(x') <_{L^*} z <_{L^*} \mathcal{N}(x)$. The last inequality is strict, because if z were equal to $\mathcal{N}(x)$, then $pr_1\mathcal{N}(x) = 0$ and from Corollary 3.2 would follow $pr_2\mathcal{N}(\mathcal{N}(x)) = 0$, i.e., $1 - x_1 = 0$, which is a contradiction since $x_1 \neq 1$ was assumed. Similarly, we obtain $\mathcal{N}(x') <_{L^*} z' <_{L^*} \mathcal{N}(x)$, since from $x' \neq 1_{L^*}$ follows that $\mathcal{N}(x') \neq 0_{L^*}$ and thus $pr_2\mathcal{N}(x') \neq 1$. On the other hand, it is easy to see that neither $z \leq_{L^*} z'$ nor $z \geq_{L^*} z'$. But, using the fact that $\mathcal{N}$ is decreasing, we obtain $pr_1\mathcal{N}(z) = pr_1\mathcal{N}(z') = x_1$. Hence, either $\mathcal{N}(z) \leq_{L^*} \mathcal{N}(z')$ or $\mathcal{N}(z) \geq_{L^*} \mathcal{N}(z')$, so either $z \leq_{L^*} z'$ or $z \geq_{L^*} z'$, which is a contradiction. Hence, our initial assumption was wrong, and $pr_2\mathcal{N}(x) = pr_2\mathcal{N}(x')$. Similarly, it is proven that $pr_1\mathcal{N}(1 - x_1, x_1) = pr_1\mathcal{N}(0, x_1)$. ∎

Corollary 3.4: Let $\mathcal{N}$ be an involutive intuitionistic fuzzy negator. For all $x = (x_1, x_2) \in L^*$, $pr_2\mathcal{N}(x) = pr_2\mathcal{N}(x_1, 1 - x_1) = pr_2\mathcal{N}(x_1, 0)$ and $pr_1\mathcal{N}(x) = pr_1\mathcal{N}(1 - x_2, x_2) = pr_1\mathcal{N}(0, x_2)$.

Lemma 3.5: Let $\mathcal{N}$ be an involutive intuitionistic fuzzy negator. Then $\mathcal{N}(D) = D$.

Proof: Let $x \in D$ and assume that $\mathcal{N}(x) \notin D$ (see Fig. 3. Then for $y = (1 - pr_2\mathcal{N}(x), pr_2\mathcal{N}(x))$, we have $y >_{L^*} \mathcal{N}(x)$ and by Corollary 3.4 $pr_1\mathcal{N}(y) = pr_1\mathcal{N}(\mathcal{N}(x)) = x_1$. Since $\mathcal{N}$ is decreasing, $\mathcal{N}(y) \leq_{L^*} x$, so $pr_2\mathcal{N}(y) = 1 - x_1$. Hence, $\mathcal{N}(y) = x$ and so, since $\mathcal{N}$ is involutive, $y = \mathcal{N}(x)$, which is a contradiction. Hence, $\mathcal{N}(D) \subseteq D$. Since $\mathcal{N}$ is involutive, it follows easily that $\mathcal{N}(D) = D$. ∎

Now, we can prove the main result of this section: any involutive intuitionistic fuzzy negator can be represented using an involutive fuzzy negator, where a fuzzy negator is defined as a decreasing $[0,1] - [0,1]$ mapping N satisfying $N(0) = 1$ and $N(1) = 0$.

Theorem 3.6: Let $\mathcal{N}$ be an involutive intuitionistic fuzzy negator, and let the $[0,1] - [0,1]$ mapping N be defined by, for $a \in [0,1]$, $N(a) = pr_1\mathcal{N}(a, 1-a)$. Then, for all $x \in L^*$, $\mathcal{N}(x) = (N(1-x_2), 1 - N(x_1))$. Moreover, N is an involutive fuzzy negator. Conversely, if N is an involutive fuzzy negator, then the $L^* - L^*$ mapping $\mathcal{N}$ defined by, for all $x \in L^*$, $\mathcal{N}(x) = (N(1-x_2), 1 - N(x_1))$ is an involutive intuitionistic fuzzy negator.

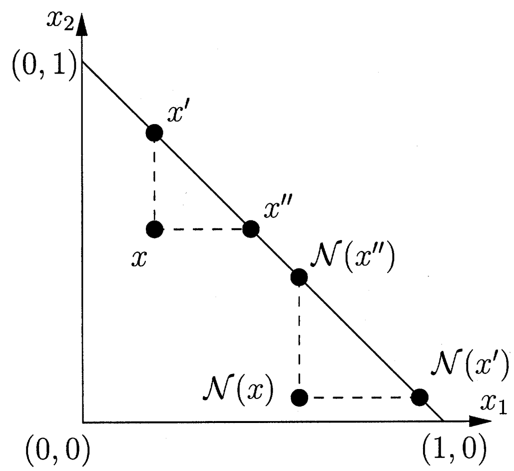

Fig. 4. Proof of Theorem 3.6.

Proof: Let N be the $[0,1]-[0,1]$ mapping defined by, for $a \in [0,1]$, $N(a) = pr_1\mathcal{N}(a, 1-a)$. Let $a, b \in [0,1]$ be such that $a \leq b$, then $(a, 1-a) \leq_{L^*} (b, 1-b)$. Since $\mathcal{N}$ is decreasing, we obtain $\mathcal{N}(a, 1-a) \geq_{L^*} \mathcal{N}(b, 1-b)$ and, thus, $N(a) \geq N(b)$. From $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$, it follows that $N(0) = 1$ and $N(1) = 0$. Let now arbitrarily $a \in [0,1]$, then $N(N(a)) = pr_1\mathcal{N}(N(a), 1-N(a))$. Since by Lemma 3.5 $\mathcal{N}(D) = D$, we have that $pr_2\mathcal{N}(a, 1-a) = 1 - pr_1\mathcal{N}(a, 1-a) = 1 - N(a)$ and, hence, $\mathcal{N}(a, 1-a) = (N(a), 1-N(a))$. Hence, $pr_1\mathcal{N}(N(a), 1-N(a)) = pr_1\mathcal{N}(\mathcal{N}(a, 1-a)) = pr_1(a, 1-a) = a$. Hence, N is an involutive fuzzy negator.

Let now x be an arbitrary element of L^*, and define $x' = (x_1, 1-x_1)$, $x'' = (1-x_2, x_2)$. From Corollary 3.4, it follows that $pr_2\mathcal{N}(x') = pr_2\mathcal{N}(x)$ and $pr_1\mathcal{N}(x'') = pr_1\mathcal{N}(x)$. Furthermore from Lemma 3.5 follows $\mathcal{N}(x') \in D$ and $\mathcal{N}(x'') \in D$. From this, it follows that $pr_1\mathcal{N}(x') = N(x_1)$, $pr_2\mathcal{N}(x') = 1 - N(x_1)$, $pr_1\mathcal{N}(x'') = N(1-x_2)$ and $pr_2\mathcal{N}(x'') = 1 - N(1-x_2)$. Hence, the result follows.

The converse was proven in [22]. ■

For instance, if $N = N_s$, where N_s denotes the standard fuzzy negator defined as, for all $x \in [0,1]$, $N_s(x) = 1-x$, then we obtain the intuitionistic fuzzy standard negator $\mathcal{N}_s$.

IV. Intuitionistic Fuzzy Triangular Norms and Conorms

Classically, a triangular norm T on [0,1] is defined as an increasing, commutative, associative $[0,1]^2-[0,1]$ mapping satisfying $T(1,x) = x$, for all $x \in [0,1]$. A triangular conorm S is defined as an increasing, commutative, associative $[0,1]^2-[0,1]$ mapping satisfying $S(0,x) = x$, for all $x \in [0,1]$. Using the lattice $(L^*, \leq_{L^*})$ these definitions can be straightforwardly extended to the intuitionistic fuzzy case.

Definition 4.1: An intuitionistic fuzzy triangular norm is an $(L^*)^2 - L^*$ mapping $\mathcal{T}$ satisfying the following conditions:

- $(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$, (border condition);
- $(\forall (x,y) \in (L^*)^2)(\mathcal{T}(x,y) = \mathcal{T}(y,x))$, (commutativity);
- $(\forall (x,y,z) \in (L^*)^3)(\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(y, \mathcal{T}(x,z)))$, (associativity);
- $(\forall (x,x',y,y') \in (L^*)^4)$ $(x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Rightarrow \mathcal{T}(x,y) \leq_{L^*} \mathcal{T}(x',y'))$. (monotonicity)

Definition 4.2: An intuitionistic fuzzy triangular conorm is an $(L^*)^2 - L^*$ mapping $\mathcal{S}$ satisfying the following conditions:

- $(\forall x \in L^*)(\mathcal{S}(x, 0_{L^*}) = x)$, (border condition);
- $(\forall (x,y) \in (L^*)^2)(\mathcal{S}(x,y) = \mathcal{S}(y,x))$, (commutativity);
- $(\forall (x,y,z) \in (L^*)^3)(\mathcal{S}(x, \mathcal{S}(y,z)) = \mathcal{S}(y, \mathcal{S}(x,z)))$, (associativity);
- $(\forall (x,x',y,y') \in (L^*)^4)$ $(x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Rightarrow \mathcal{S}(x,y) \leq_{L^*} \mathcal{S}(x',y'))$. (monotonicity).

Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm, then for any intuitionistic fuzzy negator $\mathcal{N}$, the mapping $\mathcal{T}^*$ defined by $\mathcal{T}^*(x,y) = \mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y)))$, for all $x, y \in L^*$, is an intuitionistic fuzzy t-conorm. $\mathcal{T}^*$ is called the dual intuitionistic fuzzy t-conorm of $\mathcal{T}$ w.r.t. $\mathcal{N}$. Similarly, if $\mathcal{S}$ is an intuitionistic fuzzy t-conorm, then for any intuitionistic fuzzy negator $\mathcal{N}$, the mapping $\mathcal{S}^*$ defined by $\mathcal{S}^*(x,y) = \mathcal{N}(\mathcal{S}(\mathcal{N}(x), \mathcal{N}(y)))$, for all $x, y \in L^*$ is an intuitionistic fuzzy t-norm, called the dual intuitionistic fuzzy t-norm of $\mathcal{S}$ w.r.t. $\mathcal{N}$.

Some examples of intuitionistic fuzzy t-norms and t-conorms are, for $x, y \in L^*$

i) $\inf(x,y) = (\min(x_1,y_1), \max(x_2,y_2))$;
ii) $\sup(x,y) = (\max(x_1,y_1), \min(x_2,y_2))$;
iii) $\mathcal{T}(x,y) = (\max(0, x_1+y_1-1), \min(1, x_2+y_2))$;
iv) $\mathcal{T}(x,y) = (x_1y_1, x_2+y_2-x_2y_2)$.

In fuzzy set theory t-norms are used to model the intersection of two fuzzy sets, or equivalently in logical terms, conjunction. t-conorms are used to model disjunction or union. Also in intuitionistic fuzzy set theory union and intersection can be modeled by the newly defined intuitionistic fuzzy t-norms and t-conorms. We define, for all $u \in U$ and A, B intuitionistic fuzzy sets in U

$$A \cap_{\mathcal{T}} B(u) = \mathcal{T}(A(u), B(u))$$
$$A \cup_{\mathcal{S}} B(u) = \mathcal{S}(A(u), B(u)).$$

Intuitionistic fuzzy t-norms and t-conorms can be constructed using t-norms and t-conorms on [0,1] in the following way. Let T be a t-norm and S a t-conorm, then the dual t-norm S^* of S is defined by $S^*(x,y) = 1 - S(1-x, 1-y)$, for all $x, y \in [0,1]$. If $T \leq S^*$, i.e., if for all $x, y \in [0,1]$, $T(x,y) \leq S^*(x,y)$, then the mapping $\mathcal{T}$ defined by $\mathcal{T}(x,y) = (T(x_1,y_1), S(x_2,y_2))$, for all $x, y \in L^*$, is an intuitionistic fuzzy t-norm, and the mapping $\mathcal{S}$ defined by $\mathcal{S}(x,y) = (S(x_1,y_1), T(x_2,y_2))$, for all $x, y \in L^*$, is an intuitionistic fuzzy t-conorm. Note that the condition $T \leq S^*$ is necessary and sufficient for $\mathcal{T}(x,y)$ and $\mathcal{S}(x,y)$ to be elements of L^* for all $x, y \in L^*$. We write $\mathcal{T} = (T,S)$ and $\mathcal{S} = (S,T)$.

Unfortunately, the converse is not always true. It is not possible to find for any intuitionistic fuzzy t-norm $\mathcal{T}$ a t-norm T and a t-conorm S such that $\mathcal{T} = (T,S)$. Consider for instance the intuitionistic fuzzy t-norm $\mathcal{T}_W$ given by, for all x, y in L^* [23]

$$\mathcal{T}_W(x,y) = (\max(0, x_1+y_1-1), \min(1, x_2+1-y_1, y_2+1-x_1)).$$

It is easily verified that $\mathcal{T}_W(1_{L^*}, y) = y$, $\mathcal{T}_W$ is commutative and increasing. We check the associativity

$$\begin{aligned}\mathcal{T}_W&(x, \mathcal{T}_W(y,z))\\ &= (\max(0, x_1 + \max(0, y_1 + z_1 - 1) - 1),\\ &\quad \min(1, x_2 + 1 - \max(0, y_1 + z_1 - 1),\\ &\quad\quad \min(1, y_2 + 1 - z_1, z_2 + 1 - y_1) + 1 - x_1))\\ &= (\max(0, x_1 + y_1 + z_1 - 2),\\ &\quad \min(1, x_2 + 1 - y_1 + 1 - z_1, y_2 + 1 - z_1 + 1 - x_1,\\ &\quad\quad z_2 + 1 - y_1 + 1 - x_1))\end{aligned}$$

which is symmetrical in x and y and, thus, equal to $\mathcal{T}_W(y, \mathcal{T}_W(x,z))$. Hence, $\mathcal{T}_W$ is indeed an intuitionistic fuzzy t-norm.

Let $x = (0.5, 0.3)$, $x' = (0.3, 0.3)$ and $y = (0.2, 0)$. Then, $pr_2\mathcal{T}_W(x,y) = 0.5 \neq pr_2\mathcal{T}_W(x',y) = 0.7$. Hence, there exist no T and S such that $\mathcal{T}_W = (T, S)$, since this would imply that $pr_2\mathcal{T}_W(x,y)$ is independent from x_1. In the sequel we will call this intuitionistic fuzzy t-norm the intuitionistic fuzzy Łukasiewicz t-norm.

To distinguish between these two kinds of intuitionistic fuzzy t-norms, we introduce the notion of t-representability [24].

Definition 4.3: An intuitionistic fuzzy t-norm $\mathcal{T}$ is called t-representable iff there exist a t-norm T and a t-conorm S on [0,1] such that, for all $x, y \in L^*$

$$\mathcal{T}(x,y) = (T(x_1,y_1), S(x_2,y_2)).$$

An intuitionistic fuzzy t-conorm $\mathcal{S}$ is called t-representable iff there exist a t-norm T and a t-conorm S on [0,1] such that, for all $x, y \in L^*$

$$\mathcal{S}(x,y) = (S(x_1,y_1), T(x_2,y_2)).$$

We have the following theorem assuring the t-representability of the dual of a given t-representable intuitionistic fuzzy t-norm or t-conorm.

Theorem 4.1: The dual intuitionistic fuzzy t-norm with respect to an involutive negator $\mathcal{N}$ on L^* of a t-representable intuitionistic fuzzy t-conorm is t-representable. The dual intuitionistic fuzzy t-conorm with respect to an involutive negator $\mathcal{N}$ on L^* of a t-representable intuitionistic fuzzy t-norm is t-representable.

Proof: Let $\mathcal{S}$ be a t-representable intuitionistic fuzzy t-conorm, i.e., there exists a t-norm T and a fuzzy t-conorm S on [0,1] such that, for all x, $y \in L^*$, $\mathcal{S}(x,y) = (S(x_1,y_1), T(x_2,y_2))$. Then, by Theorem 3.6, for any involutive negator $\mathcal{N}$ on L^*, there exists an involutive negator N on [0,1] such that $\mathcal{N}(x_1,x_2) = (N(1-x_2), 1 - N(x_1))$. Hence

$$\begin{aligned}\mathcal{S}^*(x,y) &= \mathcal{N}(\mathcal{S}(\mathcal{N}(x), \mathcal{N}(y)))\\ &= \mathcal{N}(S(N(1-x_2), N(1-y_2)),\\ &\quad\quad T(1 - N(x_1), 1 - N(y_1)))\\ &= (N(1 - T(1 - N(x_1), 1 - N(y_1))),\\ &\quad\quad 1 - N(S(N(1-x_2), N(1-y_2))))\\ &= ((N \circ N_s \circ T \circ ((N_s \circ N) \times (N_s \circ N)))(x_1, y_1),\\ &\quad\quad (N_s \circ N \circ S \circ ((N \circ N_s) \times (N \circ N_s)))(x_2, y_2)).\end{aligned}$$

It is verified straightforwardly that $N \circ N_s \circ T \circ ((N_s \circ N) \times (N_s \circ N))$ is a t-norm on [0,1] and $N_s \circ N \circ S \circ ((N \circ N_s) \times (N \circ N_s))$ is a t-conorm. Hence, $\mathcal{S}^*$ is t-representable. Similarly, it is proven that the dual of a t-representable intuitionistic fuzzy t-norm is t-representable. ■

As a corollary we also have that if an intuitionistic fuzzy t-norm $\mathcal{T}$ is not t-representable, then the dual $\mathcal{T}^*$ is not t-representable either, since otherwise $\mathcal{T}$ would be t-representable as the dual of the t-representable intuitionistic fuzzy t-conorm $\mathcal{T}^*$. Similarly, the dual of a non t-representable intuitionistic fuzzy t-conorm is not t-representable.

V. Intuitionistic Fuzzy Continuity

A metric space (M, d) is a pair formed by a nonempty set M and a $M \times M - \mathbb{R}$ mapping d which satisfies the following properties [25]:

- $(\forall (x,y) \in M^2)(d(x,y) \geq 0)$ (nonnegativity);
- $(\forall (x,y) \in M^2)(x = y \Leftrightarrow d(x,y) = 0)$ (separation);
- $(\forall (x,y) \in M^2)(d(x,y) = d(y,x))$ (symmetry);
- $(\forall (x,y,z) \in M^3)(d(x,z) \leq d(x,y) + d(y,z))$ (triangle inequality).

A mapping d satisfying these properties is called a metric or a distance function on M.

Well-known metrics include the Euclidean distance and the Hamming distance. In the two-dimensional space $\mathbb{R}^2$, they are defined as follows.

- The Euclidean distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ is given by
$$d^E(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$
- The Hamming distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ is given by
$$d^H(x,y) = |x_1 - y_1| + |x_2 - y_2|.$$

If we restrict these distances to L^* then we obtain the metric spaces (L^*, d^E) and (L^*, d^H).

Denote for any $x \in L^*$, $x_\pi = 1 - x_1 - x_2$. Szmidt and Kacprzyk [26] have defined two distances on L^* based on the Euclidean and the Hamming distance, where also x_π is used.

- The intuitionistic fuzzy Euclidean distance $d^E_{L^*}$ between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in L^* is given by
$$d^E_{L^*}(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_\pi - y_\pi)^2}.$$
- The intuitionistic fuzzy Hamming distance $d^H_{L^*}$ between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in L^* is given by
$$d^H_{L^*}(x,y) = |x_1 - y_1| + |x_2 - y_2| + |x_\pi - y_\pi|.$$

We now prove that these four distances are topologically equivalent, i.e., the class of open sets in the metric space generated by the distances is the same in all four cases. So, continuity w.r.t. one of these metric spaces is equivalent to continuity w.r.t. another of them.

Denote by $B(a, \varepsilon)$ the open ball with center a and radius ε defined as $B(a, \varepsilon) = \{x | x \in M \text{ and } d(x,a) < \varepsilon\}$. If for a set $A \subseteq M$ holds that for any $a \in A$, there exists an $\varepsilon > 0$ such that $B(a,\varepsilon) \subseteq A$, then A is called open in (M, d). Denote by τ_d the class of all open sets in (M, d) [25].

Theorem 5.1: Let $d_1, d_2 \in \{d^E, d^H, d^E_{L^*}, d^H_{L^*}\}$, then $\tau_{d_1} = \tau_{d_2}$.

Proof: We prove, for instance, that $\tau_{d^E} = \tau_{d^E_{L^*}}$. It is necessary and sufficient to prove that the open balls in (L^*, d^E) are open in $(L^*, d^E_{L^*})$ and conversely [25].

The open ball $B(a,\varepsilon)$ in (L^*, d^E) is the set

$$B(a,\varepsilon) = \left\{x | x \in L^* \text{ and } \sqrt{(x_1-a_1)^2+(x_2-a_2)^2} < \varepsilon\right\}.$$

The open balls in $(L^*, d^E_{L^*})$ have the form

$$B'(a,\varepsilon) = \{x | x \in L^* \text{ and } \sqrt{(x_1-a_1)^2+(x_2-a_2)^2+(x_\pi-a_\pi)^2} < \varepsilon\Big\}.$$

Let y be an arbitrary element of $B(a,\varepsilon)$, then $\varepsilon_2 = d^E(y,a) < \varepsilon$. Let $\varepsilon_3 = \varepsilon - \varepsilon_2$. Then for any $z \in B'(y,\varepsilon_3)$ holds that $d^E(z,y) \le d^E_{L^*}(z,y) < \varepsilon_3$. From the triangle inequality follows $d^E(z,a) \le d^E(z,y)+d^E(y,a) < \varepsilon_3+\varepsilon_2 = \varepsilon$. Hence, for any element $y \in B(a,\varepsilon)$, the open ball $B'(y,\varepsilon_3)$ is a subset of $B(a,\varepsilon)$. Hence, $B(a,\varepsilon) \in \tau_{d^E_{L^*}}$.

In general, we have $x_\pi - a_\pi = (1-x_1-x_2)-(1-a_1-a_2) = a_1-x_1+a_2-x_2$. Hence, $(x_\pi-a_\pi)^2 = (x_1-a_1)^2+(x_2-a_2)^2+2(x_1-a_1)(x_2-a_2)$. If $d^E(x,a) < \varepsilon$, then $(x_1-a_1)^2+(x_2-a_2)^2 < \varepsilon^2$. Moreover, $2(x_1-a_1)(x_2-a_2) \le 2\max\{(x_1-a_1)^2,(x_2-a_2)^2\} < 2\varepsilon^2$. This yields $(x_\pi-a_\pi)^2 < 3\varepsilon^2$, and so $(x_1-a_1)^2+(x_2-a_2)^2+(x_\pi-a_\pi)^2 < 4\varepsilon^2$, thus $d^E_{L^*}(x,a) < 2\varepsilon$.

Let now y be an arbitrary element of $B'(a,\varepsilon)$, then $\varepsilon_2 = d^E_{L^*}(y,a) < \varepsilon$. Let $\varepsilon_3 = (1/2)(\varepsilon-\varepsilon_2)$. Then, for all $z \in B(y,\varepsilon_3)$ holds that $d^E(z,y) < \varepsilon_3$, so $d^E_{L^*}(z,y) < 2\varepsilon_3$. From the triangle inequality follows: $d^E_{L^*}(z,a)^E \le d^E_{L^*}(z,y)+d^E_{L^*}(y,a) < 2\varepsilon_3+\varepsilon_2 = \varepsilon$. So, for any $y \in B'(a,\varepsilon)$, there exists an open ball with center y which is contained in $B'(a,\varepsilon)$. Hence, $B^\varepsilon(a,\varepsilon) \in \tau_{d^E}$. It follows that $\tau_{d^E} = \tau_{d^E_{L^*}}$. ■

From now on, if we talk about continuity in L^*, then we will mean continuity w.r.t. one of the four metric spaces defined previously. We define intuitionistic fuzzy left- and right-continuity for further usage.

Definition 5.1: Let F be an arbitrary L^*-L^* mapping and $a \in L^*$, then F is called intuitionistic fuzzy left-continuous in a iff

$$(\forall\varepsilon>0)(\exists\delta_1>0)(\exists\delta_2>0)(\forall x \in L^*)\,(a_1-\delta_1 < x_1 \le a_1 \text{ and } a_2 \le x_2 < a_2+\delta_2 \Rightarrow d(F(x),F(a)) < \varepsilon). \quad (1)$$

Let F be an arbitrary L^*-L^* mapping and $a \in L^*$, then F is called intuitionistic fuzzy right-continuous in a iff

$$(\forall\varepsilon>0)(\exists\delta_1>0)(\exists\delta_2>0)(\forall x \in L^*)\,(a_1 \le x_1 < a_1+\delta_1 \text{ and } a_2-\delta_2 < x_2 \le a_2 \Rightarrow d(F(x),F(a)) < \varepsilon). \quad (2)$$

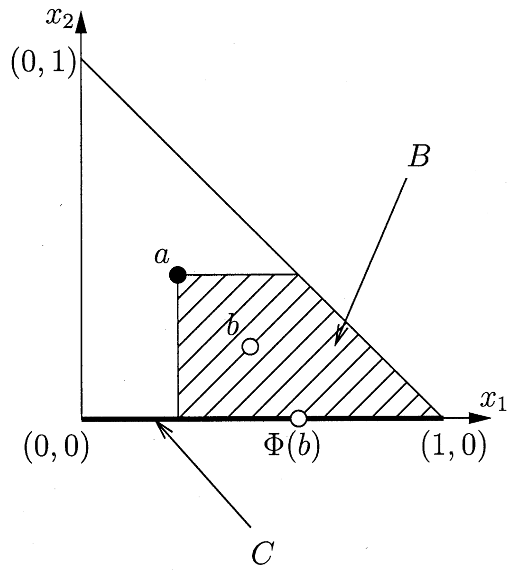

Fig. 5. Sets $B = \{x | x \in L^* \text{ and } x \ge_{L^*} a\} \setminus \{b\}$ and $C = \{x | x \in L^* \text{ and } x \ge_{L^*} (0,0)\} \setminus \{\Phi(b)\}$.

Let F be an arbitrary L^*-L^* mapping, then F is called intuitionistic fuzzy left-continuous (respectively, right-continuous) iff F is intuitionistic fuzzy left-continuous (respectively, right-continuous) in every point of L^*.

VI. Continuous Increasing Permutations of L^*

Definition 6.1: [27] Let (L^*, d) be one of the metric spaces defined in Section V and let A be a nonempty subset of L^*. If for any two open sets B and C in the metric space (A, d_A), where d_A denotes the restriction of d to A, holds that $B \cup C = A$ implies either $B = \emptyset$ or $C = \emptyset$, then A is called connected.

Lemma 6.1: Let Φ be a continuous increasing permutation of L^*. Then $\Phi(0,0) = (0,0)$.

Proof: Assume $\Phi(a) = (0,0)$, where $a \ne (0,0)$. Let first $a_2 > 0$. Then, for all $x \ge_{L^*} a$, $\Phi(x) \ge_{L^*} (0,0)$. Let arbitrarily $b \in L^*$ such that $a_1 < b_1$, $0 < b_2 < a_2$ and $b \notin D$, and consider the set $B = \{x | x \in L^* \text{ and } x \ge_{L^*} a\} \setminus \{b\}$ and the set $C = \{x | x \in L^* \text{ and } x \ge_{L^*} (0,0)\} \setminus \{\Phi(b)\}$ (see Fig. 5). Then $\Phi(B) \subseteq C$. Clearly B is connected and C is not. Now, we have $\Phi(a) = (0,0) <_{L^*} \Phi(b)$ and $\Phi(1_{L^*}) = 1_{L^*} >_{L^*} \Phi(b)$, since Φ is an increasing permutation of L^*. Hence, $\Phi(B)$ is not connected. Since connectedness is a continuous invariant (see, e.g., [27]), we obtain a contradiction, so $a_2 = 0$.

Assume now that $a_1 > 0$. Let now $b \le_{L^*} a$ and consider the sets $B = \{x | x \in L^* \text{ and } x \le_{L^*} a\} \setminus \{b\}$ and $C = \{x | x \in L^* \text{ and } x \le_{L^*} (0,0)\} \setminus \{\Phi(b)\}$. In a similar way as before, a contradiction is obtained.

From this, it follows that, for all $a \in L^*$ such that $a \ne (0,0)$, $\Phi(a) \ne (0,0)$. Since Φ is a permutation of L^*, it follows that $\Phi(0,0) = (0,0)$. ■

Corollary 6.2: Let Φ be a continuous increasing permutation of L^*. Then, for all $a \in [0,1]$, $pr_2\Phi(a,0) = 0$ and $pr_1\Phi(0,a) = 0$.

Proof: Let $a \in [0,1]$. Then $(a,0) \ge_{L^*} (0,0)$. Since Φ is increasing, we obtain $\Phi(a,0) \ge_{L^*} \Phi(0,0) = (0,0)$. Hence, $pr_2\Phi(a,0) = 0$. Similarly we obtain $pr_1\Phi(0,a) = 0$. ■

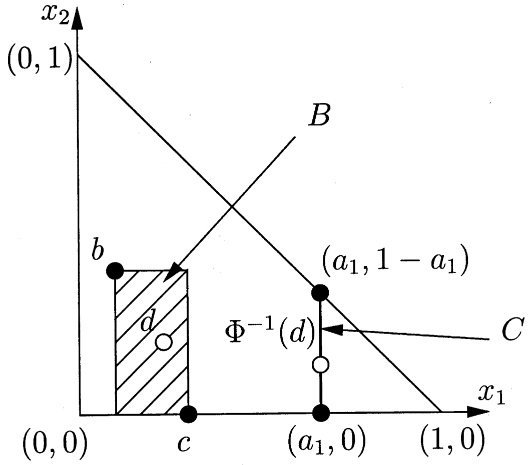

Fig. 6. Sets $B = \{x | x \in L^*$ and $b \leq_{L^*} x \leq_{L^*} c\} \setminus \{d\}$ and $C = \{x | x \in L^*$ and $x_1 = a_1\} \setminus \{\Phi^{-1}(d)\}$.

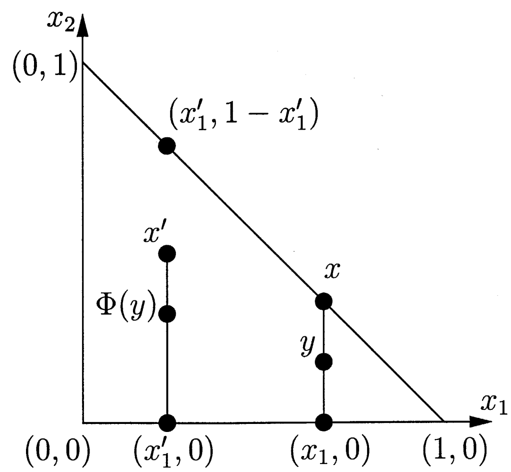

Fig. 7. Proof of Lemma 6.7.

Definition 6.2: [25] A subset A of the metric space ($\mathbb{R}^2$, d) is said to be closed if, whenever $(x_n)_{n\in\mathbb{N}^*}$ is a sequence in A which converges to a limit in $\mathbb{R}^2$, then the limit belongs to A.

Definition 6.3: [25] A subset A of the metric space ($\mathbb{R}^2$, d) is said to be bounded if there is an $a \in \mathbb{R}^2$ and an $M \in]0,+\infty[$ such that $d(a,x) \leq M$, for all $x \in A$.

Definition 6.4: [25] A subset A of the metric space ($\mathbb{R}^2$, d) is said to be sequentially compact if every sequence $(x_n)_{n\in\mathbb{N}^*}$ in A has a subsequence $(x_{n_k})_{k\in\mathbb{N}^*}$ which converges to a limit in A.

Lemma 6.3: [25] A subset of the metrict space ($\mathbb{R}^2$, d) is sequentially compact if and only if it is closed and bounded.

Since L^* is a closed and bounded subset of $\mathbb{R}^2$, L^* is sequentially compact.

Theorem 6.4: [25] Let $f : X \to Y$ be a continuous bijective mapping from (X, d) to (Y, d), where X, $Y \subseteq \mathbb{R}^2$. Suppose further that X is sequentially compact. Then the mapping $f^{-1} : Y \to X$ is also continuous.

Lemma 6.5: Let Φ be a continuous increasing permutation of L^* with increasing inverse. Then, for all $a_1 \in [0,1]$, $pr_1\Phi(a_1, 1-a_1) = pr_1\Phi(a_1,0)$ and $pr_2\Phi(1-a_1,a_1) = pr_2\Phi(0,a_1)$.

Proof: Since Φ is a continuous permutation of L^* and L^* is sequentially compact, Φ^{-1} is continuous. If $a_1 = 1$, then the result trivially holds. So, let arbitrarily $a_1 \in [0,1[$ and let $b = \Phi(a_1, 1-a_1)$ and $c = \Phi(a_1,0)$ and assume that $b_1 < c_1$ (see Fig. 6). Then, $\Phi^{-1}(b) = (a_1, 1-a_1)$ and $\Phi^{-1}(c) = (a_1,0)$. Let $d \in L^*$ be such that $b_1 < d_1 < c_1$, $b_2 > d_2 > c_2$ and $d \notin D$ (note that from Corollary 6.2 follows that $c_2 = 0$ and that similarly as in Lemma 6.1 can be proven that $b_2 > 0$). Then, since Φ is increasing and, hence, Φ^{-1} is increasing, $(a_1, 1-a_1) \leq_{L^*} \Phi^{-1}(d) \leq_{L^*} (a_1,0)$. Let $B = \{x | x \in L^*$ and $b \leq_{L^*} x \leq_{L^*} c\} \setminus \{d\}$ and $C = \{x | x \in L^*$ and $x_1 = a_1\} \setminus \{\Phi^{-1}(d)\}$. Then $\Phi^{-1}(B) \subseteq C$, since Φ^{-1} is increasing and bijective. Moreover, $\Phi^{-1}(b) = (a_1, 1-a_1)$ and $\Phi^{-1}(c) = (a_1,0)$, so clearly $\Phi^{-1}(B)$ is not connected. Since B is connected, we obtain a contradiction. ■ ■

Corollary 6.6: Let Φ be a continuous increasing permutation of L^* with increasing inverse. Then, for all $a \in L^*$, $pr_1\Phi(a) = pr_1\Phi(a_1, 1-a_1) = pr_1\Phi(a_1,0)$ and $pr_2\Phi(a) = pr_2\Phi(1-a_2,a_2) = pr_2\Phi(0,a_2)$.

Lemma 6.7: Let Φ be a continuous increasing permutation of L^* with increasing inverse. Then $\Phi(D) = D$.

Proof: Assume there exists an $x \in D$ such that $\Phi(x) \notin D$. From Corollary 6.6, it follows that $pr_1\Phi(x) = pr_1\Phi(x_1,0)$. Since Φ is a permutation of L^* and since from Corollary 6.2 follows $pr_2\Phi(x_1,0) = pr_2\Phi(z_1,0) = 0$, we have $pr_1\Phi(x_1,0) \neq pr_1\Phi(z_1,0) \Leftrightarrow x_1 \neq z_1$, for all $z_1 \in [0,1]$. Hence, for all $z \in L^*$, $pr_1\Phi(z) \neq pr_1\Phi(x) \Leftrightarrow z_1 \neq x_1$.

Let now $x' = \Phi(x)$ (see Fig. 7). Since $\Phi(x) \notin D$, we have $x'_2 < 1 - x'_1$. Now, since Φ is increasing, it follows that for all $y \in L^*$ such that $y_1 = x_1$ (and, thus, $y_2 \leq x_2 = 1 - x_1$), $\Phi(x) \leq_{L^*} \Phi(y)$. Since from the above we know that, for all $y \in L^*$ such that $y_1 \neq x_1$, $pr_1\Phi(y) \neq x'_1$, and now we have that for all $y \in L^*$ such that $y_1 = x_1$, $(x'_1, 1-x'_1) <_{L^*} x' = \Phi(x) \leq_{L^*} \Phi(y)$, we obtain that $(x'_1, 1-x'_1) \notin \Phi(L^*)$. This is a contradiction to the fact that Φ is bijective. Hence, $\Phi(D) \subseteq D$.

Since Φ is bijective and increasing, we have $\Phi(0_{L^*}) = 0_{L^*}$ and $\Phi(1_{L^*}) = 1_{L^*}$. Since Φ is continuous and D is connected, $\Phi(D)$ should be connected. From $\Phi(D) \subseteq D$, $0_{L^*} \in \Phi(D)$, and $1_{L^*} \in \Phi(D)$ follows now that $\Phi(D) = D$. ■

Theorem 6.8: Let Φ be a continuous increasing permutation of L^* with increasing inverse. Then, there exists a continuous increasing permutation φ of [0,1] such that, for all $x \in L^*$

$$\Phi(x) = (\varphi(x_1), 1 - \varphi(1 - x_2))\,. \tag{3}$$

Conversely, for any increasing permutation φ of [0,1], the $L^* - L^*$ mapping Φ defined by (3) is an increasing permutation of L^*.

Proof: Let x be an arbitrary element of L^*, and let $x' = (x_1, 1-x_1)$ and $x'' = (1-x_2, x_2)$ (see Fig. 8). Then, $pr_1\Phi(x) = pr_1\Phi(x')$ and $pr_2\Phi(x) = pr_2\Phi(x'')$. Define $\varphi(a) = pr_1\Phi(a, 1-a)$, for all $a \in [0,1]$. Then, for all a, $b \in [0,1]$, $\varphi(a) \neq \varphi(b)$ is equivalent to $\Phi(a, 1-a) \neq \Phi(b, 1-b)$, since $\Phi(D) = D$. Since Φ is a permutation of L^*, this is equivalent to $(a, 1-a) \neq (b, 1-b)$, which is equivalent to $a \neq b$. Hence, φ is injective. Clearly,

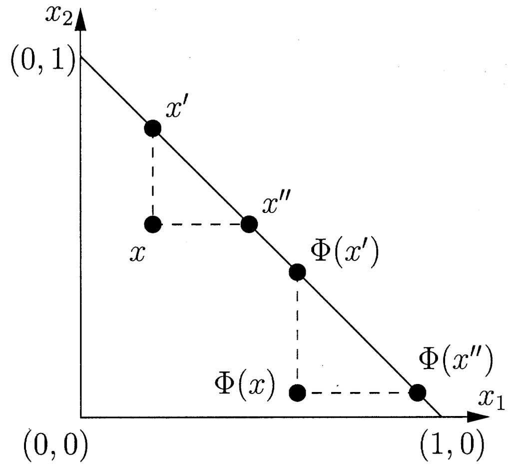

Fig. 8. Proof of Theorem 6.8.

φ is surjective, because $\Phi(D) = D$. Hence, φ is a permutation of [0,1]. φ is increasing, since Φ is increasing. From $\varphi = pr_1 \circ \Phi \circ (id_{[0,1]}, N_s)$, where $id_{[0,1]}$ denotes the identical function on [0,1], and the fact that all these functions are continuous, follows the continuity of φ.

From this, it follows that $pr_1\Phi(x) = pr_1\Phi(x') = \varphi(x_1)$ and $pr_2\Phi(x) = pr_2\Phi(x'') = pr_2(\varphi(1-x_2), 1-\varphi(1-x_2)) = 1-\varphi(1-x_2)$. ■

VII. Residuation Principle

We say that an intuitionistic fuzzy t-norm $\mathcal{T}$ satisfies the residuation principle if and only if, for all $x, y, z \in L^*$

$$\mathcal{T}(x,z) \leq_{L^*} y \Leftrightarrow z \leq_{L^*} \mathcal{I}_{\mathcal{T}}(x,y) \tag{4}$$

where $\mathcal{I}_{\mathcal{T}}$ denotes the residual implicator generated by $\mathcal{T}$, defined as

$$\mathcal{I}_{\mathcal{T}}(x,y) = \sup\{\gamma | \gamma \in L^* \text{ and } \mathcal{T}(x,\gamma) \leq_{L^*} y\}.$$

Theorem 7.1 (Characterization of Supremum in L^*): Let A be an arbitrary nonempty subset of L^* and $a \in L^*$. Then, $a = \sup A$ if and only if

$$(\forall x \in A)\,(x \leq_{L^*} a) \quad \text{and} \quad (\forall \varepsilon_1 > 0)(\exists z \in A)(z_1 > a_1 - \varepsilon_1) \quad \text{and} \quad (\forall \varepsilon_2 > 0)(\exists z \in A)(z_2 < a_2 + \varepsilon_2). \tag{5}$$

Proof: Assume $a = \sup A$, i.e.,

$$(\forall x \in A)\,(x \leq_{L^*} a) \quad \text{and} \quad (\forall b \in L^*)\,((\forall x \in A)\,(x \leq_{L^*} b) \Rightarrow a \leq_{L^*} b).$$

Assume that (5) does not hold, i.e.,

$$(\exists \varepsilon_1 > 0)(\forall z \in A)(z_1 \leq a_1 - \varepsilon_1) \quad \text{or} \quad (\exists \varepsilon_2 > 0)(\forall z' \in A)\,(z'_2 \geq a_2 + \varepsilon_2).$$

If $(\exists \varepsilon_1 > 0)(\forall z \in A)(z_1 \leq a_1 - \varepsilon_1)$, let then $b = (a_1 - \varepsilon_1, a_2)$, then clearly $b <_{L^*} a$ and $(\forall z \in A)(z \leq_{L^*} b)$, since $z_1 \leq a_1 - \varepsilon_1$ and $z_2 \geq a_2$. This contradicts the fact that $a = \sup A$. If on the other hand $(\exists \varepsilon_2 > 0)(\forall z' \in A)(z'_2 \geq a_2 + \varepsilon_2)$, let then $b = (\min(a_1, 1-a_2-\varepsilon_2), a_2+\varepsilon_2)$. Then, clearly, $b <_{L^*} a$ and for all $z \in A$, $z_2 \geq a_2 + \varepsilon_2$ and $z_1 \leq a_1$. If $z_1 > 1 - a_2 - \varepsilon_2$, then $z_1 > 1 - a_2 - \varepsilon_2 \geq 1 - z_2$, which is a contradiction. Hence, $z_1 \leq 1 - a_2 - \varepsilon_2$, so $z_1 \leq \min(a_1, 1-a_2-\varepsilon_2)$. Thus, we obtain that for all $z \in A$, $z \leq_{L^*} b$. However, we had $b <_{L^*} a$, which is a contradiction. Hence, (5) holds.

Assume conversely that (5) holds for $a \in L^*$, but that a is not the supremum of A. Then

$$(\exists b \in L^*)\,((\forall x \in A)\,(x \leq_{L^*} b) \text{ and } a \not\leq_{L^*} b).$$

Then, either $a_1 > b_1$ or $a_2 < b_2$. Assume $\varepsilon = a_1 - b_1 > 0$. Then for all $z \in A$, $z \leq_{L^*} b$ implies $z_1 \leq b_1 = a_1 - \varepsilon$, which contradicts (5). If on the other hand $\varepsilon = b_2 - a_2 > 0$, then for all $z \in A$, $z \leq_{L^*} b$ implies $z_2 \geq b_2 = a_2 + \varepsilon$, which again is a contradiction. Hence, $a = \sup A$. ■

In a similar way, the following characterization of infimum is proven.

Theorem 7.2 (Characterization of Infimum in L^*): Let A be an arbitrary nonempty subset of L^* and $a \in L^*$. Then, $a = \inf A$ if and only if

$$(\forall x \in A)\,(x \geq_{L^*} a) \quad \text{and} \quad (\forall \varepsilon_1 > 0)(\exists z \in A)(z_1 < a_1 + \varepsilon_1) \quad \text{and} \quad (\forall \varepsilon_2 > 0)(\exists z \in A)(z_2 > a_2 - \varepsilon_2). \tag{6}$$

Note that the following expression is equivalent to (5):

$$a = \sup A \Leftrightarrow (\forall x \in A)\,(x \leq_{L^*} a) \text{ and } (\forall \varepsilon > 0)\,((\exists z \in A)(z_1 > a_1 - \varepsilon) \text{ and } (\exists z \in A)(z_2 < a_2 + \varepsilon)).$$

Theorem 7.3: Let f be any increasing $L^* - L^*$ mapping. If

$$\sup f(Z) = f(\sup Z) \tag{7}$$

for all nonempty subsets Z of L^*, then f is intuitionistic fuzzy left-continuous.

Proof: Assume that (7) holds and let z^* be an arbitrary element of L^*. Then, we must prove

$$(\forall \varepsilon > 0)(\exists \delta_1 > 0)(\exists \delta_2 > 0)(\forall z \in L^*)\,(z_1^* - \delta_1 < z_1 \leq z_1^* \text{ and } z_2^* \leq z_2 < z_2^* + \delta_2 \Rightarrow |pr_1 f(z) - pr_1 f(z^*)| + |pr_2 f(z) - pr_2 f(z^*)| < \varepsilon).$$

Let now $Z = \{z | z \in L^* \text{ and } z_1 < z_1^* \text{ and } z_2 > z_2^*\}$, then $z^* = \sup Z$. Let $a = f(\sup_{z \in Z} z)$ or taking (7) into account: $\sup_{z \in Z} f(z) = a$.

From the characterization of supremum in L^*, we obtain

$$(\forall z \in Z)\,(f(z) \leq_{L^*} a) \text{ and } (\forall \varepsilon > 0)\,((\exists z \in Z)(pr_1 f(z) > a_1 - \varepsilon) \text{ and } (\exists z' \in Z)\,(pr_2 f(z') < a_2 + \varepsilon)).$$

Let $\varepsilon > 0$ be arbitrary. Then there exist $z, z' \in Z$ such that $a_1 \geq pr_1 f(z) > a_1 - (\varepsilon/2)$ and $a_2 \leq pr_2 f(z') < a_2 + (\varepsilon/2)$. Since $z_1 < z_1^*$, $z_2 > z_2^*$, $z'_1 < z_1^*$ and $z'_2 > z_2^*$, we have $\max(z_1, z'_1) < z_1^*$ and $\min(z_2, z'_2) > z_2^*$, so $\sup(z, z') \in Z$. Hence, since f is increasing, we obtain $a_1 \geq pr_1 f(\sup(z, z^*)) \geq pr_1 f(z) > a_1 - (\varepsilon/2)$ and $a_2 \leq pr_2 f(\sup(z, z')) \leq pr_2 f(z') < a_2 + (\varepsilon/2)$.

Let now $\delta_1 = |z_1^* - \max(z_1, z'_1)|$ and $\delta_2 = |z_2^* - \min(z_2, z'_2)|$). Then, for all $x \in L^*$ such that $z_1^* - \delta_1 < x_1 \leq z_1^*$ and $z_2^* \leq x_2 < z_2^* + \delta_2$, we have $x >_{L^*} \sup(z, z')$. Since f is increasing, it follows that $f(\sup(z, z')) \leq_{L^*} f(x) \leq_{L^*} f(z^*) = a$. Hence, $a_1 \geq pr_1 f(x) > a_1 - (\varepsilon/2)$ and $a_2 \leq pr_2 f(x) < a_2 + (\varepsilon/2)$, so $|pr_1 f(x) - a_1| + |pr_2 f(x) - a_2| < \varepsilon$, what we had to prove. ■

Similarly, the following theorem is proven.

Theorem 7.4: Let f be any increasing $L^* - L^*$ mapping. If

$$\inf f(Z) = f(\inf Z) \tag{8}$$

for all nonempty subsets Z of L^*, then f is intuitionistic fuzzy right continuous.

Theorem 7.5: Let f be any increasing $L^* - L^*$ mapping such that $pr_1 f(x)$ is independent of x_2 and $pr_2 f(x)$ is independent of x_1. If f is intuitionistic fuzzy left continuous, then

$$\sup f(Z) = f(\sup Z)$$

for all nonempty subsets Z of L^*.

Proof: First note that for an arbitrary nonempty set Z and for arbitrary $z \in Z$, $f(z) \leq_{L^*} f(\sup_{z \in Z} z)$, hence, $\sup_{z \in Z} f(z) \leq_{L^*} f(\sup_{z \in Z} z)$.

Denote by z^* the supremum of Z, i.e. $z^* = \sup Z$, and let $f(x) = (f_1(x_1), f_2(x_2))$, for all $x \in L^*$. Let $\varepsilon > 0$. Then by intuitionistic fuzzy left-continuity of f there exist δ_1 and δ_2 such that (1) holds.

From the characterization of supremum in L^* follows that there exist x and x' in Z such that $z_1^* \geq x_1 > z_1^* - \delta_1$ and $z_2^* \leq x'_2 < z_2^* + \delta_2$. Since $x \leq_{L^*} z^*$ and $x' \leq_{L^*} z^*$, we have $\sup(x, x') \leq_{L^*} z^*$. Hence, $z_1^* \geq \max(x_1, x'_1) \geq x_1 > z_1^* - \delta_1$ and $z_2^* \leq \min(x_2, x'_2) \leq x_2 < z_2^* + \delta_2$. From the intuitionistic fuzzy left-continuity of f follows that $|f_1(\max(x_1, x'_1)) - f_1(z_1^*)| + |f_2(\min(x_2, x'_2)) - f_2(z_2^*)| < \varepsilon$. Hence, $|f_1(\max(x_1, x'_1)) - f_1(z_1^*)| < \varepsilon$ and $|f_2(\min(x_2, x'_2)) - f_2(z_2^*)| < \varepsilon$. Since f is increasing, we obtain $0 \leq f_1(z_1^*) - f_1(\max(x_1, x'_1)) < \varepsilon$ and $0 \leq f_2(\min(x_2, x'_2)) - f_2(z_2^*) < \varepsilon$.

If $\max(x_1, x'_1) = x_1$, then there exists a $z \in Z$ such that $f_1(z_1) > f_1(z_1^*) - \varepsilon$, namely $z = x$. If $\max(x_1, x'_1) = x'_1$, then x' satisfies this condition. Similarly there exists a z' such that $f_2(z_2) < f_2(z_2^*) + \varepsilon$. From the characterization of supremum in L^* follows that $\sup_{z \in Z} f(z) = f(z^*) = f(\sup_{z \in Z} z)$. ■

Theorem 7.6: Let f be any increasing $L^* - L^*$ mapping such that $pr_1 f(x)$ is independent of x_2 and $pr_2 f(x)$ is independent of x_1. If f is intuitionistic fuzzy right-continuous, then

$$\inf f(Z) = f(\inf Z)$$

for all nonempty subsets Z of L^*.

Theorem 7.7: Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm. Then, $\mathcal{T}$ satisfies the residuation principle if and only if

$$\sup_{z \in Z} \mathcal{T}(x, z) = \mathcal{T}\left(x, \sup_{z \in Z} z\right) \tag{9}$$

for any $x \in L^*$ and any nonempty subset Z of L^*.

Proof: Assume first that (9) holds for any $x \in L^*$ and any $Z \subseteq L^*$. If $\mathcal{T}(x, z) \leq_{L^*} y$, then evidently $z \in \{\gamma | \gamma \in L^*$ and $\mathcal{T}(x, \gamma) \leq_{L^*} y\}$, so that $\mathcal{I}_{\mathcal{T}}(x, y) \geq_{L^*} z$. If, conversely, $\mathcal{I}_{\mathcal{T}}(x, y) \geq_{L^*} z$, then

$$\begin{aligned}
\mathcal{T}(x, z) &\leq_{L^*} \mathcal{T}(x, \mathcal{I}_{\mathcal{T}}(x, y)) \quad (\mathcal{T}(x, .) \text{ is increasing})\\
&= \mathcal{T}(x, \sup\{\gamma | \gamma \in L^* \text{ and } \mathcal{T}(x, \gamma) \leq_{L^*} y\})\\
&\qquad (\text{definition } \mathcal{I}_{\mathcal{T}})\\
&= \sup\{\mathcal{T}(x, \gamma) | \gamma \in L^* \text{ and } \mathcal{T}(x, \gamma) \leq_{L^*} y\}\\
&\qquad (\mathcal{T} \text{ satisfies (9)})\\
&\leq_{L^*} y.
\end{aligned}$$

Assume now that $\mathcal{T}$ satisfies the residuation principle. Consider an arbitrary $x \in L^*$ and $Z \subseteq L^*$, and let

$$y = \sup_{z \in Z} \mathcal{T}(x, z) \quad z^* = \sup Z.$$

Hence, we obtain $(\forall z \in Z)(\mathcal{T}(x, z) \leq_{L^*} y)$, so

$$(\forall z \in Z)(z \in \{\gamma | \gamma \in L^* \text{ and } \mathcal{T}(x, \gamma) \leq_{L^*} y\})$$

i.e., $Z \subseteq \{\gamma | \gamma \in L^*$ and $\mathcal{T}(x, \gamma) \leq_{L^*} y\}$. Thus $z^* = \sup Z \leq_{L^*} \sup\{\gamma | \gamma \in L^*$and $\mathcal{T}(x, \gamma) \leq_{L^*} y\} = \mathcal{I}_{\mathcal{T}}(x, y)$. From the residuation principle follows now that $\mathcal{T}(x, z^*) \leq_{L^*} y$, i.e.,

$$\mathcal{T}(x, \sup Z) \leq_{L^*} \sup_{z \in Z} \mathcal{T}(x, z).$$

Since for all $z \in Z$, $\mathcal{T}(x, z) \leq_{L^*} \mathcal{T}(x, \sup Z)$, and so $\sup_{z \in Z} \mathcal{T}(x, z) \leq_{L^*} \mathcal{T}(x, \sup Z)$, we obtain the required equality. ■

Theorem 7.8: Let $\mathcal{T}$ be a t-representable intuitionistic fuzzy t-norm. Then, the partial mappings of $\mathcal{T}$ are intuitionistic fuzzy left-continuous if and only if $\mathcal{T}$ satisfies the residuation principle.

Proof: Let $\mathcal{T}$ be a t-representable intuitionistic fuzzy t-norm with intuitionistic fuzzy left-continuous partial mappings and let x be an arbitrary element of L^*, then $pr_1 \mathcal{T}(x, y)$ is independent of y_2 and $pr_2 \mathcal{T}(x, y)$ is independent of y_1. From Theorem 7.5, it follows that for any $Z \subseteq L^*$

$$\sup_{z \in Z} \mathcal{T}(x, z) = \mathcal{T}\left(x, \sup_{z \in Z} z\right).$$

From Theorem 7.7, it follows now that $\mathcal{T}$ satisfies the residuation principle.

Conversely, if $\mathcal{T}$ satisfies the residuation principle, then by Theorems 7.3 and 7.7 follows that $\mathcal{T}$ is intuitionistic fuzzy left-continuous. ■

Counterexample 7.1: In general from intuitionistic fuzzy left-continuity it cannot be deduced that an intuitionistic

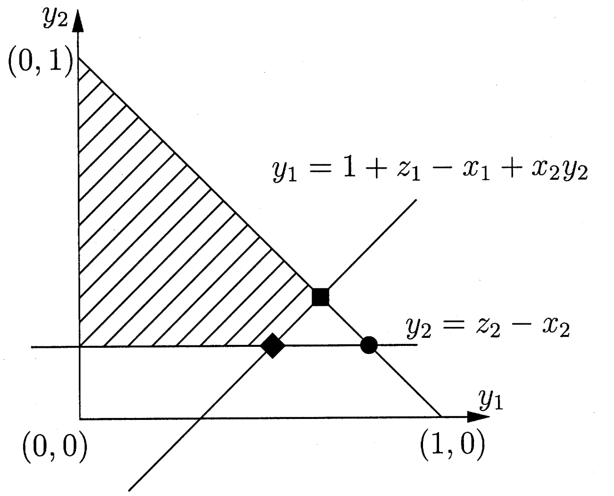

Fig. 9. Lines $y_1 = 1 + z_1 - x_1 + x_2y_2$ and $y_2 = z_2 - x_2$ (first situation).

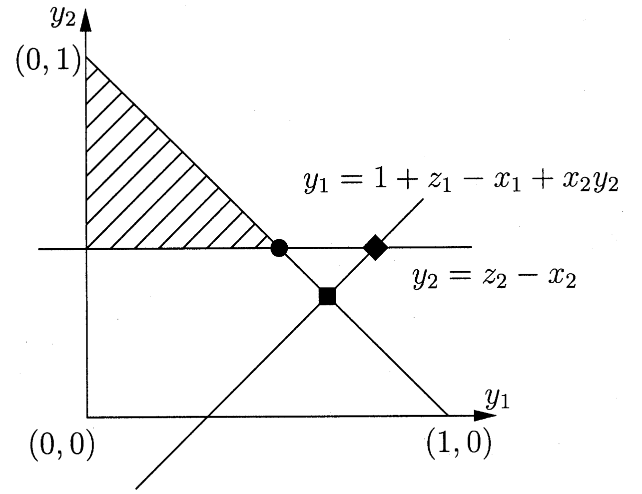

Fig. 10. Lines $y_1 = 1+z_1-x_1+x_2y_2$ and $y_2 = z_2-x_2$ (second situation).

fuzzy t-norm satisfies the residuation principle. Consider for instance the $(L^*)^2 - L^*$ mapping $\mathcal{T}$ defined as $\mathcal{T}(x,y) = (\max(0, x_1 + y_1 - x_2y_2 - 1), \min(1, x_2 + y_2))$, for all $x, y \in L^*$. Then, $\mathcal{T}(x, 1_{L^*}) = x$, $\mathcal{T}$ is commutative and increasing. We also obtain

$$\begin{aligned}\mathcal{T}\left(x, \mathcal{T}(y,z)\right) = &\left(\max\left(0, x_1 + \max(0, y_1 + z_1 - y_2z_2 - 1)\right.\right.\\ &\left. - x_2 \min(1, y_2 + z_2) - 1\right),\\ &\left.\min(1, x_2 + \min(1, y_2 + z_2))\right)\\ = &\left(\max\left(0, x_1 + y_1 + z_1 - y_2z_2 - 1\right.\right.\\ &\left. - x_2 \min(1, y_2 + z_2) - 1\right),\\ &\left.\min(1, x_2 + y_2 + z_2)\right).\end{aligned}$$

If $\min(1, y_2 + z_2) = 1$, i.e., $y_2 + z_2 \geq 1$, then $y_1 + z_1 \leq 1$. We then have $x_1 + y_1 + z_1 - 2 - y_2z_2 - x_2 \min(1, y_2 + z_2) = x_1 + y_1 + z_1 - 2 - y_2z_2 - x_2 \geq 0$ if and only if $x_1 + y_1 + z_1 \geq 2 + y_2z_2 + x_2$. But $y_1 + z_1 \leq 1$, so we have $x_1 + y_1 + z_1 \leq 1 + x_1$. If $x_1 = 1$, then $x = 1_{L^*}$ and $\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(y,z) = \mathcal{T}(y, \mathcal{T}(x,z))$. So, assume $x_1 < 1$. Then $1 + x_1 < 2$. Thus $2 > x_1 + y_1 + z_1 \geq 2 + y_2z_2 + x_2$, which is a contradiction. Hence, $x_1 + y_1 + z_1 - 2 - y_2z_2 - x_2 < 0$, and so $\mathcal{T}(x, \mathcal{T}(y,z)) = (0, \min(1, x_2 + y_2 + z_2)) = \mathcal{T}(y, \mathcal{T}(x,z))$. On the other hand, if $\min(1, y_2 + z_2) = y_2 + z_2$, then we obtain

$$\begin{aligned}\mathcal{T}\left(x, \mathcal{T}(y,z)\right) = &\left(\max(0, x_1 + y_1 + z_1 - y_2z_2 - 1\right.\\ &\left. - x_2y_2 - x_2z_2 - 1),\right.\\ &\left.\min(1, x_2 + y_2 + z_2)\right)\\ = &\,\mathcal{T}\left(y, \mathcal{T}(x,z)\right).\end{aligned}$$

Clearly, $\mathcal{T}$ is continuous.

We obtain the following:

$$\begin{aligned}&\mathcal{T}(x,y) \leq_{L^*} z\\ &\quad\Leftrightarrow x_1 + y_1 - x_2y_2 - 1 \leq z_1 \text{ and } x_2 + y_2 \geq z_2\\ &\quad\Leftrightarrow \begin{cases} y_1 - x_2y_2 \leq 1 + z_1 - x_1\\ y_2 \geq z_2 - x_2\\ y_1 + y_2 \leq 1\end{cases}.\end{aligned}$$

Four possible situations are drawn in Figs. 9–12; the shaded area denotes the set of elements $y \in L^*$ such that $\mathcal{T}(x,y) \leq_{L^*} z$.

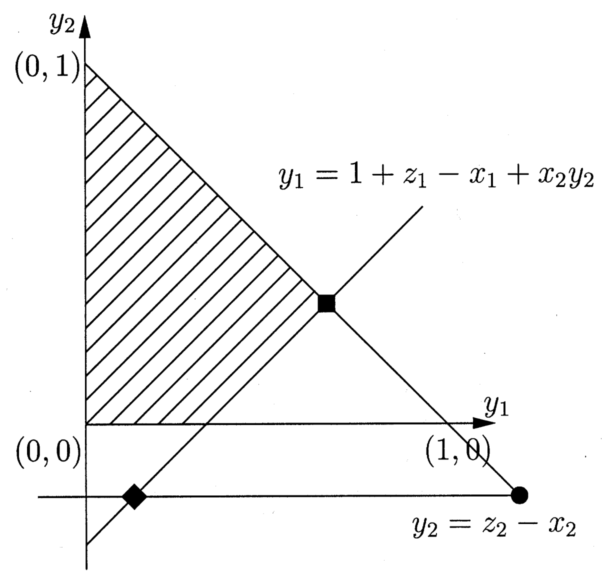

Fig. 11. Lines $y_1 = 1+z_1-x_1+x_2y_2$ and $y_2 = z_2-x_2$ (third situation).

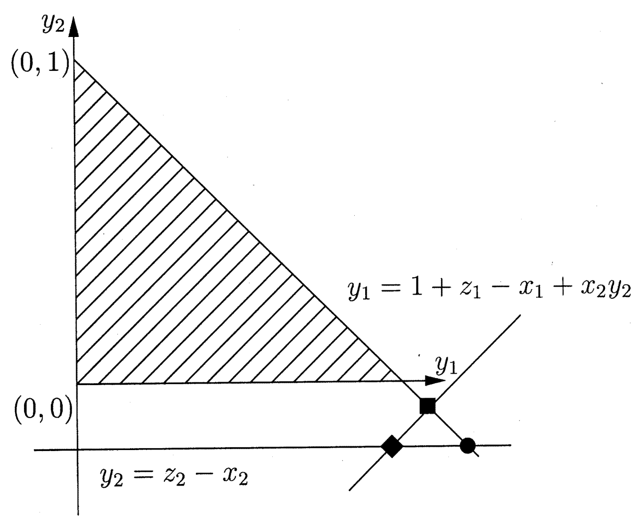

Fig. 12. Lines $y_1 = 1+z_1-x_1+x_2y_2$ and $y_2 = z_2-x_2$ (fourth situation).

The intersection of the lines $y_1 = 1 + z_1 - x_1 + x_2 y_2$ and $y_2 = z_2 - x_2$ is given by $(1+z_1-x_1+x_2(z_2-x_2), z_2-x_2)$, while the intersection of the lines $y_1 = 1+z_1-x_1+x_2y_2$ and $y_1+y_2 = 1$ is given by $(1 + ((z_1 - x_1)/(1 + x_2)), ((x_1 - z_1)/(1 + x_2)))$. So, in Fig. 9,

$$\begin{aligned}\sup\{y|y\in L^* \text{ and } \mathcal{T}(x,y)\le_{L^*} z\} &= \sup\left\{(1+z_1-x_1+x_2(z_2-x_2), z_2-x_2),\left(1+\frac{z_1-x_1}{1+x_2}, \frac{x_1-z_1}{1+x_2}\right)\right\}\\ &= \left(\max\left(1+z_1-x_1+x_2(z_2-x_2), 1+\frac{z_1-x_1}{1+x_2}\right),\min\left(z_2-x_2, \frac{x_1-z_1}{1+x_2}\right)\right)\\ &= \left(\max\left(1+z_1-x_1+x_2(z_2-x_2), 1+\frac{z_1-x_1}{1+x_2}\right), z_2-x_2\right).\end{aligned}$$

The intersection of the lines $y_2 = z_2 - x_2$ and $y_1 + y_2 = 1$ is given by $(1 + x_2 - z_2, z_2 - x_2)$, which is the supremum of $\{y|y \in L^* \text{ and } \mathcal{T}(x, y) \le_{L^*} z\}$ in Fig. 10.

In the third situation, we have that $z_2 - x_2 < 0$, so we have to consider the intersection of $y_1 = 1 + z_1 - x_1 + x_2y_2$ and $y_2 = 0$, i.e., $(1 + z_1 - x_1, 0)$, and then we obtain

$$\sup\{y|y\in L^* \text{ and } \mathcal{T}(x,y)\le_{L^*} z\} = \left(\max\left(1+z_1-x_1, 1+\frac{z_1-x_1}{1+x_2}\right), 0\right).$$

In the fourth situation, we also have that for all y on the line $y_1 = 1 + z_1 - x_1 + x_2y_2$, $y_1 + y_2 > 1$. So in this case, the supremum of $\{y|y \in L^* \text{ and } \mathcal{T}(x, y) \le_{L^*} z\}$ is equal to 1_{L^*}.

In all of the four situations, we get that

$$\begin{aligned}\mathcal{I}_\mathcal{T}(x,z) &= \sup\{y|y\in L^* \text{ and } \mathcal{T}(x,y)\le_{L^*} z\}\\ &= \Big(\min\Big(1, 1+x_2-z_2, \max\Big(1+z_1-x_1+x_2\max(0,z_2-x_2), 1+\frac{z_1-x_1}{1+x_2}\Big)\Big), \max(0,z_2-x_2)\Big).\end{aligned}$$

Let now $x = (0.5, 0.4)$ and $y = (0.3, 0.5)$. Then $z = \mathcal{I}_\mathcal{T}(x, y) = ((6/7), 0.1) = (0.857142\ldots, 0.1)$, but $\mathcal{T}(x, z) = (0.317142\ldots, 0.5) \not\le_{L^*} y$. Hence, $\mathcal{T}$ is a continuous intuitionistic fuzzy t-norm which does not satisfy the residuation principle.

VIII. Representation of Intuitionistic Fuzzy T-Norms

Definition 8.1: An intuitionistic fuzzy t-norm $\mathcal{T}$ is called Archimedean if and only if, for all $x \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$, $\mathcal{T}(x, x) <_{L^*} x$.

Definition 8.2: An intuitionistic fuzzy t-norm $\mathcal{T}$ is called nilpotent if and only if there exist $x, y \in L^* \setminus \{0_{L^*}\}$ such that $\mathcal{T}(x, y) = 0_{L^*}$.

An intuitionistic fuzzy t-norm $\mathcal{T}$ is called intuitionistic fuzzy nilpotent if and only if there exist $x, y \in L^*$ such that x_1 and y_1 are nonzero and such that $\mathcal{T}(x, y) = 0_{L^*}$.

Lemma 8.1: Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm satisfying the residuation principle. Then, for any x, y, z in L^* such that $\mathcal{T}(x, y) = z$, there exists an $y' \in L^*$ such that $y' \ge_{L^*} y$ and

$$\mathcal{T}(x, y') = z \quad \text{and} \quad y' = \mathcal{I}_\mathcal{T}(x, z). \tag{10}$$

Proof: Let x, y and z be elements of L^* for which $\mathcal{T}(x, y) = z$. Then from the residuation principle follows that $\mathcal{I}_\mathcal{T}(x, z) \ge_{L^*} y$. Define y' as $y' = \mathcal{I}_\mathcal{T}(x, z)$, then clearly $y' \ge_{L^*} y$. Since $\mathcal{T}$ is increasing, it follows that $\mathcal{T}(x, y') \ge_{L^*} \mathcal{T}(x, y)$, i.e. $\mathcal{T}(x, y') \ge_{L^*} z$.

On the other hand, since $\mathcal{I}_\mathcal{T}(x, z) = y'$, from the residuation principle follows that $\mathcal{T}(x, y') \le_{L^*} z$. Hence, $\mathcal{T}(x, y') = z$, thus (10) holds. ■

Lemma 8.2: Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm satisfying the residuation principle, and x, y and y' arbitrary elements of L^*. Assume y and y' satisfy

$$\begin{aligned}\mathcal{T}(x, y) = z \quad &\text{and} \quad \mathcal{I}_\mathcal{T}(x, z) = y\\ \mathcal{T}(x, y') = z' \quad &\text{and} \quad \mathcal{I}_\mathcal{T}(x, z') = y'.\end{aligned}$$

Then $z \le_{L^*} z' \Leftrightarrow y \le_{L^*} y'$.

Proof: Assume x, y and y' satisfy the previous conditions. If $z = \mathcal{T}(x, y) \le_{L^*} z' = \mathcal{T}(x, y')$, then, since $\mathcal{I}_\mathcal{T}$ is increasing in its second component, $y = \mathcal{I}_\mathcal{T}(x, z) \le_{L^*} \mathcal{I}_\mathcal{T}(x, z') = y'$, so $y \le_{L^*} y'$. Conversely, from $y \le_{L^*} y'$ follows immediately $z \le_{L^*} z'$, since $\mathcal{T}$ is increasing. ■

Lemma 8.3: Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm satisfying the residuation principle. Then, for any x, $y \in L^*$, $pr_1\mathcal{T}(x, y)$ is independent of x_2 and y_2.

Proof: Let arbitrarily $y \in L^*$, then $\sup\{(0, 0), y\} = (y_1, 0)$. Since $\mathcal{T}$ satisfies the residuation principle and using Theorem 7.7, it follows that $\mathcal{T}(x, (y_1, 0)) = \sup\{\mathcal{T}(x, (0, 0)), \mathcal{T}(x, y)\}$, so $pr_1\mathcal{T}(x, (y_1, 0)) = \max\{pr_1\mathcal{T}(x, (0, 0)), pr_1\mathcal{T}(x, y)\} = \max\{0, pr_1\mathcal{T}(x, y)\} = pr_1\mathcal{T}(x, y)$. Hence, $pr_1\mathcal{T}(x, y)$ is independent of y_2. Since $\mathcal{T}$ is commutative, it follows that $pr_1\mathcal{T}(x, y)$ is also independent of x_2. ■

Lemma 8.4: Let $\mathcal{T}$ be a continuous Archimedean intuitionistic fuzzy nilpotent intuitionistic fuzzy t-norm satisfying the residuation principle. Then the mapping T, defined by $T(x_1, y_1) = pr_1\mathcal{T}((x_1, 0), (y_1, 0))$, for all $x_1, y_1 \in [0, 1]$, is a continuous Archimedean nilpotent t-norm.

Proof: From Lemma 8.3 it follows that $pr_1\mathcal{T}(x, y)$ is independent of x_2 and y_2. So let T be the $[0, 1]^2 - [0, 1]$ mapping defined by $T(x_1, y_1) = pr_1\mathcal{T}((x_1, 0), (y_1, 0))$, for all x_1, $y_1 \in [0, 1]$. Then, for all x_1, $y_1 \in [0, 1]$

$$T(x_1, y_1) = pr_1\mathcal{T}((x_1, x_2), (y_1, y_2)), \quad \forall x_2 \in [0, 1 - x_1], \forall y_2 \in [0, 1 - y_1]. \tag{11}$$

Since $\mathcal{T}(x, 1_{L^*}) = x$, for all $x \in L^*$, we have $T(x_1, 1) = x_1$, for all $x_1 \in [0, 1]$. Let x_1, y_1 and z_1 be arbitrary elements of [0,1]. Then, since $\mathcal{T}$ is associative,

$\mathcal{T}((x_1,0),\mathcal{T}((y_1,0),(z_1,0))) = \mathcal{T}(\mathcal{T}((x_1,0),(y_1,0)),(z_1,0))$. Hence

$$\begin{aligned} T(x_1, T(y_1,z_1)) &= pr_1\mathcal{T}((x_1,0),(pr_1\mathcal{T}((y_1,0),(z_1,0)),0)) \\ &= pr_1\mathcal{T}((x_1,0),(pr_1\mathcal{T}((y_1,0),(z_1,0)), \\ &\qquad pr_2\mathcal{T}((y_1,0),(z_1,0)))) \text{ (using (11))} \\ &= pr_1\mathcal{T}((x_1,0),\mathcal{T}((y_1,0),(z_1,0))) \\ &= pr_1\mathcal{T}(\mathcal{T}((x_1,0),(y_1,0)),(z_1,0)) \\ &= T(T(x_1,y_1),z_1). \end{aligned}$$

From the fact that $\mathcal{T}$ is commutative and increasing follows easily that T is commutative and increasing. Hence, T is a t-norm on [0,1].

Now we prove that T is Archimedean. Let $x = (x_1, 1-x_1) \in L^*$. Since $\mathcal{T}$ is Archimedean, we have that $\mathcal{T}(x,x) <_{L^*} x$. Hence, $pr_1\mathcal{T}(x,x) < x_1$, otherwise we would have $pr_2\mathcal{T}(x,x) > x_2 = 1 - x_1$, which is impossible if $pr_1\mathcal{T}(x,x) = x_1$, since $\mathcal{T}(x,x) \in L^*$. Since $pr_1\mathcal{T}(x,x) < x_1$ holds for all $x = (x_1, 1-x_1)$, we obtain $T(x_1,x_1) < x_1$, for all $x_1 \in [0,1]$. Hence, T is Archimedean.

Since $\mathcal{T}$ is intuitionistic fuzzy nilpotent, there exist $x, y \in L^*$ such that $x_1 > 0$, $y_1 > 0$ and $\mathcal{T}(x,y) = 0_{L^*}$, and thus $T(x_1,y_1) = 0$. Hence, T is nilpotent. ■

Theorem 8.5: Let $\mathcal{T}$ be a continuous Archimedean intuitionistic fuzzy nilpotent intuitionistic fuzzy t-norm satisfying the residuation principle. Then there exists an increasing permutation φ of [0,1] such that, for any $x, y \in L^*$,

$$pr_1\mathcal{T}(x,y) = \varphi^{-1}(\max(0, \phi(x_1) + \phi(y_1) - 1)).$$

Proof: This follows immediately from Lemma 8.4 and the corresponding representation theorem for a continuous Archimedean nilpotent t-norm on [0,1]. ■

Lemma 8.6: Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm satisfying the residuation principle. Assume $\mathcal{I}_\mathcal{T}(D,D) \subseteq D$. Then for all $x \in D$ and $y_2 \in [0,1]$,

$$pr_2\mathcal{T}(x,(0,y_2)) = pr_2\mathcal{T}(x,(1-y_2,y_2)).$$

Proof: Let $x, z \in D$ and $y \in L^*$, then from the residuation principle follows $\mathcal{T}(x,y) \leq_{L^*} z \Leftrightarrow y \leq_{L^*} y' = \mathcal{I}_\mathcal{T}(x,z)$. From $x, z \in D$ follows that $y' \in D$. Now we have $\mathcal{T}(x,y) \leq_{L^*} z$ if and only if $pr_2\mathcal{T}(x,y) \geq z_2$, since $z \in D$, and similarly $y \leq_{L^*} y'$ if and only if $y_2 \geq y'_2$. So, we obtain

$$pr_2\mathcal{T}(x,y) \geq z_2 \Leftrightarrow y_2 \geq y'_2. \tag{12}$$

Let now for arbitrary $y_2 \in [0,1]$, $pr_2\mathcal{T}(x,(0,y_2)) = z_2$, then from (12) follows $y_2 \geq y'_2 = pr_2\mathcal{I}_\mathcal{T}(x,(1-z_2,z_2))$. From (12) follows also $pr_2\mathcal{T}(x,(1-y_2,y_2)) \geq z_2$ since $y_2 \geq y'_2$. Since $\mathcal{T}$ is increasing, we have $pr_2\mathcal{T}(x,(1-y_2,y_2)) \leq pr_2\mathcal{T}(x,(0,y_2)) = z_2$. Hence, $pr_2\mathcal{T}(x,(0,y_2)) = pr_2\mathcal{T}(x,(1-y_2,y_2))$. ■

Lemma 8.7: Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm satisfying the residuation principle and $\mathcal{T}((0,0),(0,0)) = 0_{L^*}$. Then, for all $x_1, y_2 \in [0,1]$

$$pr_2\mathcal{T}((0,y_2),(x_1,0)) = pr_2\mathcal{T}((0,y_2),(x_1,1-x_1)).$$

Proof: Let arbitrarily $x_1, y_2 \in [0,1]$. Since $\mathcal{T}$ satisfies the residuation principle and using Theorem 7.7 and the fact that $\sup\{(0,0),(x_1,1-x_1)\} = (x_1,0)$, it follows that $pr_2\mathcal{T}((0,y_2),(x_1,0)) = \min\{pr_2\mathcal{T}((0,y_2),(0,0)), pr_2\mathcal{T}((0,y_2),(x_1,1-x_1))\}$. From $\mathcal{T}((0,0),(0,0)) = 0_{L^*}$ and the fact that $\mathcal{T}$ is increasing, it follows that $pr_2\mathcal{T}((0,y_2),(0,0)) = 1$ and, thus, $pr_2\mathcal{T}((0,y_2),(x_1,0)) = pr_2\mathcal{T}((0,y_2),(x_1,1-x_1))$. ■

Theorem 8.8: Let $\mathcal{T}$ be a continuous Archimedean intuitionistic fuzzy nilpotent intuitionistic fuzzy t-norm satisfying the residuation principle, $\mathcal{I}_\mathcal{T}(D,D) \subseteq D$ and $\mathcal{T}((0,0),(0,0)) = 0_{L^*}$. Then for all $x, y \in L^*$,

$$\begin{aligned} pr_2\mathcal{T}(x,y) = 1 - \varphi^{-1}(\max(0, &\varphi(x_1) + \varphi(1-y_2) - 1, \\ &\varphi(y_1) + \varphi(1-x_2) - 1)) \end{aligned}$$

where φ is the permutation from Theorem 8.5.

Proof: Define the $[0,1]^2 - [0,1]$ mapping f by $f(y_2,x_1) = pr_2\mathcal{T}((0,y_2),(x_1,0))$, for all $x_1, y_2 \in [0,1]$. We will first prove that $pr_2\mathcal{T}(x,y) = \min\{f(y_2,x_1), f(x_2,y_1)\}$, for all $x, y \in L^*$.

From the residuation principle, Theorem 7.7 and the fact that $\sup\{(0,y_2),(y_1,1-y_1)\} = (y_1,y_2)$ follows that $pr_2\mathcal{T}(x,y) = \min\{pr_2\mathcal{T}(x,(0,y_2)), pr_2\mathcal{T}(x,(y_1,1-y_1))\}$, for all $x, y \in L^*$. From Lemma 8.7 follows that $pr_2\mathcal{T}(x,(0,y_2)) = pr_2\mathcal{T}((x_1,0),(0,y_2)) = f(y_2,x_1)$. From Lemma 8.6 follows that $pr_2\mathcal{T}(x,(y_1,1-y_1)) = pr_2\mathcal{T}((1-x_2,x_2),(y_1,1-y_1)) = pr_2\mathcal{T}((0,x_2),(y_1,1-y_1))$ and from Lemma 8.7 follows $pr_2\mathcal{T}((0,x_2),(y_1,1-y_1)) = pr_2\mathcal{T}((0,x_2),(y_1,0)) = f(x_2,y_1)$. Hence, $pr_2\mathcal{T}(x,y) = \min\{f(y_2,x_1), f(x_2,y_1)\}$.

Let now $x \in D$ and $y \in L^*$, then from Lemma 8.6 follows $pr_2\mathcal{T}(x,y) = pr_2\mathcal{T}(x,(1-y_2,y_2)) = pr_2\mathcal{T}(x,(0,y_2))$. From Lemma 8.7 follows $pr_2\mathcal{T}(x,(0,y_2)) = pr_2\mathcal{T}((x_1,0),(0,y_2)) = f(y_2,x_1)$. So for all $x \in D$ and $y \in L^*$, we have $pr_2\mathcal{T}(x,y) = f(y_2,x_1)$. Let now also $y \in D$. Then similarly $pr_2\mathcal{T}(x,y) = f(x_2,y_1)$. So for $x, y \in D$, we obtain $pr_2\mathcal{T}(x,y) = f(1-x_1,y_1) = f(1-y_1,x_1)$. From the last equality follows that for all $x_1, y_1 \in [0,1]$, $f(x_1,y_1) = f(1-y_1,1-x_1)$. So $f(0,y_1) = f(1-y_1,1) = pr_2\mathcal{T}((1,0),(0,1-y_1)) = 1-y_1$, for all $y_1 \in [0,1]$.

Let T be the mapping defined in Lemma 8.4, $x, z \in L^*$ and $y_2 \in [0,1]$. We calculate $\mathcal{T}(\mathcal{T}((0,y_2),x),z) = \mathcal{T}((0,f(y_2,x_1)),z) = (0,f(f(y_2,x_1),z_1))$, using Lemma 8.7. Since $\mathcal{T}$ is associative, we obtain that $\mathcal{T}(\mathcal{T}((0,y_2),x),z) = \mathcal{T}((0,y_2),\mathcal{T}(x,z)) = \mathcal{T}((0,y_2),(T(x_1,z_1),pr_2\mathcal{T}(x,z))) = (0,f(y_2,T(x_1,z_1)))$. So we obtain $f(f(y_2,x_1),z_1) = f(y_2,T(x_1,z_1))$, for all $y_2, x_1, z_1 \in [0,1]$. Let now $y_2 = 0$, then $f(1-x_1,z_1) = f(f(0,x_1),z_1) =$

$f(0,T(x_1,z_1)) = 1 - T(x_1,z_1)$, for all $x_1,\ z_1 \in [0,1]$. Thus, $f(x_1,z_1) = 1 - T(1-x_1,z_1)$, for all $x_1, z_1 \in [0,1]$.

From this, it follows that $pr_2\mathcal{T}(x,y) = \min\{1 - T(1-y_2,x_1), 1 - T(1-x_2,y_1)\}$. Using Theorem 8.5 and $T(x_1,y_1) = pr_1\mathcal{T}(x,y)$, we obtain $pr_2\mathcal{T}(x,y) = \min\{1 - \varphi^{-1}(\max(0,\varphi(1-y_2)+\varphi(x_1)-1)), 1-\varphi^{-1}(\max(0,\varphi(1-x_2)+\varphi(y_1)-1))\}= 1-\varphi^{-1}(\max(0,\varphi(1-y_2)+\varphi(x_1)-1, \varphi(1-x_2)+\varphi(y_1)-1))$. ■

Theorem 8.9: Let $\mathcal{T}$ be an $(L^*)^2 - L^*$ mapping. Assume there exists a continuous increasing permutation Φ of L^* with-increasing inverse such that $\mathcal{T} = \Phi^{-1} \circ \mathcal{T}_W \circ (\Phi \times \Phi)$, where $\mathcal{T}_W$ denotes the intuitionistic fuzzy Łukasiewicz t-norm. Then, $\mathcal{T}$ is a continuous, Archimedean, intuitionistic fuzzy nilpotent intuitionistic fuzzy t-norm satisfying the residuation principle, $\mathcal{I}_{\mathcal{T}}(D,D) \subseteq D$ and $\mathcal{T}((0,0),(0,0)) = 0_{L^*}$.

Proof: First we prove that $\mathcal{T} = \Phi^{-1} \circ \mathcal{T}_W \circ (\Phi \times \Phi)$ is an intuitionistic fuzzy t-norm for any continuous increasing permutation Φ of L^*. Using the fact that for any increasing permutation Φ of L^* holds that $\Phi(1_{L^*}) = 1_{L^*}$, and the fact that $\mathcal{T}_W$ is an intuitionistic fuzzy t-norm, it is easily verified that $\mathcal{T}(1_{L^*},y) = y$ for all $y \in L^*$, $\mathcal{T}$ is commutative and increasing. We verify that $\mathcal{T}$ is associative

$$\begin{aligned}
&\mathcal{T}(x,\mathcal{T}(y,z)) \\
&\quad = \Phi^{-1}\left(\mathcal{T}_W\left(\Phi(x),\Phi\left(\Phi^{-1}\left(\mathcal{T}_W\left(\Phi(y),\Phi(z)\right)\right)\right)\right)\right) \\
&\quad = \Phi^{-1}\left(\mathcal{T}_W\left(\Phi(x),\mathcal{T}_W\left(\Phi(y),\Phi(z)\right)\right)\right) \\
&\quad = \Phi^{-1}\left(\mathcal{T}_W\left(\mathcal{T}_W\left(\Phi(x),\Phi(y)\right),\Phi(z)\right)\right) \\
&\quad = \mathcal{T}\left(\mathcal{T}(x,y),z\right)
\end{aligned}$$

using the associativity of $\mathcal{T}_W$. Since Φ and $\mathcal{T}_W$ are continuous, $\mathcal{T}$ is continuous.

Now, we prove that $\mathcal{T}_W$ satisfies the listed properties. We have $\mathcal{T}_W(x,x) = (\max(0,2x_1-1), \min(1,x_2+1-x_1)) <_{L^*} x$ as soon as $x \in L^* \setminus \{0_{L^*},1_{L^*}\}$. So $\mathcal{T}_W$ is Archimedean. If moreover $x \in D$ such that $x_1 \leq (1/2)$, then $\mathcal{T}_W(x,x) = 0_{L^*}$, so $\mathcal{T}_W$ is intuitionistic fuzzy nilpotent. We also have that $\mathcal{T}_W((0,0),(0,0)) = 0_{L^*}$. Concerning the residuation principle, we obtain successively, for x, y and z in L^*:

$$\begin{aligned}
&\mathcal{T}_W(x,y) \leq_{L^*} z \\
&\quad \Leftrightarrow x_1 + y_1 - 1 \leq z_1 \\
&\qquad \text{and } x_2 + 1 - y_1 \geq z_2 \text{ and } y_2 + 1 - x_1 \geq z_2 \\
&\quad \Leftrightarrow y_1 \leq \min(z_1 + 1 - x_1, x_2 + 1 - z_2) \\
&\qquad \text{and } y_2 \geq z_2 + x_1 - 1 \\
&\quad \Leftrightarrow y_1 \leq \min(1, z_1 + 1 - x_1, x_2 + 1 - z_2) \\
&\qquad \text{and } y_2 \geq \max(0, z_2 + x_1 - 1)
\end{aligned}$$

using the fact that $y \in L^*$ and, thus, $y_1 \leq 1$ and $y_2 \geq 0$. Hence, $\mathcal{I}_{\mathcal{T}_W}(x,z) = \sup\{\gamma | \gamma \in L^* \text{ and } \mathcal{T}_W(x,\gamma) \leq_{L^*} z\}= (\min(1,z_1+1-x_1,x_2+1-z_2), \max(0,z_2+x_1-1))$, on the condition that $a = (\min(1,z_1+1-x_1,x_2+1-z_2), \max(0,z_2+x_1-1)) \in L^*$. If $\min(1,z_1+1-x_1,x_2+1-z_2) = 1$, then $z_1 \geq x_1$, so $x_1 \leq 1-z_2$, hence, $\max(0,z_2+x_1-1) = 0$ and $a \in L^*$. If $\min(1,z_1+1-x_1,x_2+1-z_2) = z_1+1-x_1$, then $(z_1+1-x_1)+\max(0,z_2+x_1-1)$ is equal to either $(z_1+1-x_1)+(z_2+x_1-1) = z_1+z_2 \leq 1$ or $z_1+1-x_1+0 \leq 1$, so $a \in L^*$. Similarly, if $\min(1,z_1+1-x_1,x_2+1-z_2) = x_2+1-z_2$, then $a \in L^*$. Since $\mathcal{T}_W(x,y) \leq_{L^*} z \Leftrightarrow y \leq_{L^*} \mathcal{I}_{\mathcal{T}_W}(x,z)$, $\mathcal{T}_W$ satisfies the residuation principle. Let now $x,\ z \in D$, then $\mathcal{I}_{\mathcal{T}_W}(x,z) = (\min(1,z_1+1-x_1), \max(0,1-z_1+x_1-1)) = (\min(1,z_1+1-x_1), 1-\min(1,z_1+1-x_1)) \in D$.

Now, we prove that $\mathcal{T}$ satisfies the listed properties. From the fact that $\mathcal{T}_W$ is Archimedean follows $\mathcal{T}(x,x) = \Phi^{-1}(\mathcal{T}_W(\Phi(x),\Phi(x))) <_{L^*} \Phi^{-1}(\Phi(x)) = x$, for all $x \in L^*$ such that $\Phi(x) \neq 0_{L^*}$ and $\Phi(x) \neq 1_{L^*}$, i.e. for all $x \in L^* \setminus \{0_{L^*},1_{L^*}\}$.

Let now $x \in D$ be such that $pr_1\Phi(x) \leq (1/2)$. Such an x exists since from Lemma 6.7 follows that $\Phi(D) = D$ and since Φ is bijective. Then for $y = \Phi(x)$ it holds that $y_1 \leq (1/2)$ and $y \in D$. Then $\mathcal{T}(x,x) = \Phi^{-1}(\mathcal{T}_W(y,y)) = \Phi^{-1}(0_{L^*}) = 0_{L^*}$. Hence, $\mathcal{T}$ is intuitionistic fuzzy nilpotent.

Concerning the residuation principle, we obtain successively, for x, y and z in L^*

$$\begin{aligned}
\mathcal{T}(x,y) \leq_{L^*} z &\Leftrightarrow \Phi^{-1}\left(\mathcal{T}_W\left(\Phi(x),\Phi(y)\right)\right) \leq_{L^*} z \\
&\Leftrightarrow \mathcal{T}_W\left(\Phi(x),\Phi(y)\right) \leq_{L^*} \Phi(z) \\
&\Leftrightarrow \Phi(y) \leq_{L^*} \mathcal{I}_{\mathcal{T}_W}\left(\Phi(x),\Phi(z)\right) \\
&\Leftrightarrow y \leq_{L^*} \Phi^{-1}\left(\mathcal{I}_{\mathcal{T}_W}\left(\Phi(x),\Phi(z)\right)\right).
\end{aligned}$$

Hence, $\mathcal{I}_{\mathcal{T}} = \Phi^{-1} \circ \mathcal{I}_{\mathcal{T}_W} \circ (\Phi \times \Phi)$ and $\mathcal{T}$ satisfies the residuation principle.

Let now $x,\ z \in D$, then by Lemma 6.7 we obtain $\Phi(x)$, $\Phi(z) \in D$. From this, it follows $\mathcal{I}_{\mathcal{T}_W}(\Phi(x),\Phi(z)) \in D$. So, using the fact that $\Phi(D) = D$, we obtain that $\Phi^{-1}(\mathcal{I}_{\mathcal{T}_W}(\Phi(x),\Phi(z))) \in D$. Hence, $\mathcal{I}_{\mathcal{T}}(D,D) \subseteq D$.

From Lemma 6.1 follows $\mathcal{T}((0,0),(0,0)) = \Phi^{-1}(\mathcal{T}_W((0,0),(0,0)))= \Phi^{-1}(0_{L^*}) = 0_{L^*}$. ■

From Theorems 8.5, 8.8, and 8.9, the following theorem is obtained.

Theorem 8.10: Let $\mathcal{T}$ be an $(L^*)^2 - L^*$ mapping. Then the following are equivalent.

i) $\mathcal{T}$ is a continuous, Archimedean, intuitionistic fuzzy nilpotent intuitionistic fuzzy t-norm satisfying the residuation principle, $\mathcal{I}_{\mathcal{T}}(D,D) \subseteq D$ and $\mathcal{T}((0,0),(0,0)) = 0_{L^*}$.

ii) There exists a continuous increasing permutation φ of [0,1] such that, for all $x, y \in L^*$ with increasing inverse

$$\begin{aligned}
\mathcal{T}(x,y) = \big(&\varphi^{-1}\left(\max\left(0,\varphi(x_1)+\varphi(y_1)-1\right)\right), \\
&1-\varphi^{-1}\left(\max\left(0,\varphi(x_1)+\varphi(1-y_2)-1,\right.\right. \\
&\qquad\left.\left.\varphi(y_1)+\varphi(1-x_2)-1\right)\right)\big) \quad (13)
\end{aligned}$$

iii) There exists a continuous increasing permutation Φ of L^* such that $\mathcal{T} = \Phi^{-1} \circ \mathcal{T}_W \circ (\Phi \times \Phi)$.

Proof: From Theorem 8.5 and Theorem 8.8 we obtain that i) implies ii).

Assume now that ii) holds and let $\Phi(x) = (\varphi(x_1),(N_s \circ \varphi \circ N_s)(x_2))= (\varphi(x_1), 1-\varphi(1-x_2))$, for all $x \in L^*$. Then $y = \Phi(x) \Leftrightarrow y_1 = \varphi(x_1)$ and $y_2 = 1-\varphi(1-x_2) \Leftrightarrow x_1 = \varphi^{-1}(y_1)$ and $x_2 = 1-\varphi^{-1}(1-y_2)$, for all $y \in L^*$. Hence, $\Phi^{-1}(x) = (\varphi^{-1}(x_1), 1-\varphi^{-1}(1-x_2))= (\varphi^{-1}(x_1), (N_s \circ \varphi^{-1} \circ N_s)(x_2))$, for all $x \in L^*$.

We have $pr_2\mathcal{T}_W(x,y) = \min(1, x_2+1-y_1, y_2+1-x_1) = 1-\max(0, y_1-x_2, x_1-y_2)$ and

$$\begin{aligned} pr_2\mathcal{T}(x,y) &= 1-\varphi^{-1}(\max(0,\varphi(x_1)+\varphi(1-y_2)-1,\\ &\qquad \varphi(y_1)+\varphi(1-x_2)-))\\ &= (N_s\circ\varphi^{-1}\circ N_s)\\ &\quad (1-\max(0,\varphi(x_1)-(N_s\circ\varphi\circ N_s)(y_2),\\ &\qquad \varphi(y_1)-(N_s\circ\varphi\circ N_s)(x_2))\\ &= (N_s\circ\varphi^{-1}\circ N_s\circ pr_2\mathcal{T}_W)\\ &\quad ((\varphi(x_1),(N_s\circ\varphi\circ N_s)(x_2)),(\varphi(y_1),\\ &\qquad (N_s\circ\varphi\circ N_s)(y_2)))\\ &= (N_s\circ\varphi^{-1}\circ N_s\circ pr_2\mathcal{T}_W)(\Phi(x),\Phi(y)). \end{aligned}$$

Since $pr_1\mathcal{T}(x,y) = \varphi^{-1}(\max(0,\varphi(x_1)+\varphi(y_1)-1)) = (\varphi^{-1}\circ pr_1\mathcal{T}_W)(\Phi(x),\Phi(y))$, we obtain that $\mathcal{T}(x,y) = (\Phi^{-1}\circ\mathcal{T}_W)(\Phi(x),\Phi(y))$, for all $x,y\in L^*$. So from ii) follows iii).

The fact that from iii) follows i) is proven in Theorem 8.9. ■

Note that from the proof of Theorem 8.9 follows that $\mathcal{I}_\mathcal{T}(x,z) = (\varphi^{-1}(\min(1,\varphi(z_1)+1-\varphi(x_1), 1-\varphi(1-x_2)+\varphi(1-z_2))), 1-\varphi^{-1}(1-\max(0,\varphi(x_1)-\varphi(1-z_2))))$. Moreover $\mathcal{N}(x) = \mathcal{I}_\mathcal{T}(x,0_{L^*}) = (\varphi^{-1}(1-\varphi(1-x_2)), 1-\varphi^{-1}(1-\varphi(x_1)))$, so $\mathcal{N}(x) = (N(1-x_2), 1-N(x_1))$, with $N=\varphi^{-1}\circ N_s\circ\varphi$. It also follows that $\mathcal{T}(D,D)\subseteq D$.

Now, we give two examples of intuitionistic fuzzy t-norms to show that the conditions $\mathcal{I}_\mathcal{T}(D,D)\subseteq D$ and $\mathcal{T}((0,0),(0,0)) = 0_{L^*}$ are necessary in Theorem 8.10, i.e., these conditions do not follow from the other conditions. We know from Lemma 8.6 that the condition $\mathcal{I}_\mathcal{T}(D,D)\subseteq D$ implies that $pr_2\mathcal{T}(x,(0,y_2)) = pr_2\mathcal{T}(x,(1-y_2,y_2))$, for all $x\in D$ and for all $y_2\in[0,1]$, as soon as $\mathcal{T}$ is an intuitionistic fuzzy t-norm satisfying the residuation principle.

Example 8.1: Let $\mathcal{T}$ be the $(L^*)^2 - L^*$ mapping defined by, for all x, $y\in L^*$

$$\begin{aligned} \mathcal{T}(x,y) = (&\max(0,x_1+y_1-1), \min(1,y_2+2(1-x_1),\\ &x_2+2(1-y_1), 1-x_1+1-y_1)). \end{aligned}$$

Then, it is easily verified that $\mathcal{T}$ is a continuous, commutative, increasing mapping satisfying $\mathcal{T}(1_{L^*},y) = y$ and $\mathcal{T}((0,0),(0,0)) = 0_{L^*}$. Now we calculate, for x, y and z in L^*

$$\begin{aligned} &pr_2\mathcal{T}(x,\mathcal{T}(y,z))\\ &\quad= \min(1,\min(1,z_2+2(1-y_1),y_2+2(1-z_1),\\ &\qquad 1-y_1+1-z_1)+2(1-x_1), x_2\\ &\qquad +2(1-\max(0,y_1+z_1-1)), 1-x_1+1\\ &\qquad -\max(0,y_1+z_1-1))\\ &\quad= \min(1,z_2+2(1-y_1)+2(1-x_1), y_2+2(1-z_1)\\ &\qquad +2(1-x_1), 1-y_1+1-z_1+2(1-x_1),\\ &\qquad x_2+2(1-y_1)+2(1-z_1), 1-x_1+1-y_1\\ &\qquad +1-z_1) \end{aligned}$$

which is symmetrical in x and y, since $1-y_1+1-z_1+2(1-x_1)\geq 1-x_1+1-y_1+1-z_1$. Similarly $pr_1\mathcal{T}(x,\mathcal{T}(y,z))$ is symmetrical in x and y. Hence, $\mathcal{T}$ is associative, and $\mathcal{T}$ is an intuitionistic fuzzy t-norm.

Since $pr_1\mathcal{T}(x,x) < x_1$, for all $x\in L^*$ such that $x_1\in]0,1[$, and since for all $x\in L^*$ such that $x_1 = 0$, $pr_2\mathcal{T}(x,x) = 1$, $\mathcal{T}$ is Archimedean. Let now $0 < x_1 < (1/2)$ and $x_2 = 1-x_1$. Then $\mathcal{T}(x,x) = (\max(0,2x_1-1), \min(1,3(1-x_1),2(1-x_1))) = (0,1) = 0_{L^*}$. Hence, $\mathcal{T}$ is intuitionistic fuzzy nilpotent.

Now, we have, for x, y, and z in L^*

$$\begin{aligned} &\mathcal{T}(x,y)\leq_{L^*} z\\ &\quad\Leftrightarrow x_1+y_1-1\leq z_1 \text{ and } y_2+2(1-x_1)\geq z_2\\ &\qquad \text{and } x_2+2(1-y_1)\geq z_2 \text{ and } 1-x_1+1-y_1\geq z_2\\ &\quad\Leftrightarrow y_1\leq \min\left(1, z_1+1-x_1, 1-\frac{1}{2}(z_2-x_2),\right.\\ &\qquad\qquad 1-x_1+1-z_2)\\ &\qquad \text{and } y_2\geq\max(0,z_2-2(1-x_1)). \end{aligned}$$

Hence, $\mathcal{I}_\mathcal{T}(x,z) = (\min(1,z_1+1-x_1, 1-(1/2)(z_2-x_2), 1-x_1+1-z_2), \max(0,z_2-2(1-x_1)))$, for all x, $z\in L^*$, on the condition that $a = (\min(1,z_1+1-x_1,1-(1/2)(z_2-x_2), 1-x_1+1-z_2), \max(0,z_2-2(1-x_1)))\in L^*$. Since $\min(1,z_1+1-x_1,1-(1/2)(z_2-x_2),1-x_1+1-z_2)+0\leq 1+0 = 1$, and $\min(1,z_1+1-x_1,1-(1/2)(z_2-x_2),1-x_1+1-z_2)+z_2-2(1-x_1)\leq 1-x_1+1-z_2+z_2-2(1-x_1) = x_1\leq 1$, we have that $a\in L^*$, for all x, $z\in L^*$. From the above calculations follows immediately that $\mathcal{T}(x,y)\leq_{L^*} z$ if and only if $y\leq_{L^*}\mathcal{I}_\mathcal{T}(x,z)$, and so the residuation principle holds for $\mathcal{T}$.

Finally, we check that there exist an $x\in D$ and a $y_2\in[0,1]$ such that $pr_2\mathcal{T}(x,(0,y_2))\neq pr_2\mathcal{T}(x,(1-y_2,y_2))$. Let $x = (0.9, 0.1)$ and $y_2 = 0.3$. Then $pr_2\mathcal{T}(x,(0,y_2)) = \min(1,y_2+2x_2)) = 0.5\neq pr_2\mathcal{T}(x,(1-y_2,y_2)) = \min(1,y_2+2x_2,x_2+2y_2,x_2+y_2) = 0.4$.

Clearly, $\mathcal{T}$ is an intuitionistic fuzzy t-norm satisfying all the conditions of Theorem 8.10 except $\mathcal{I}_\mathcal{T}(D,D)\subseteq D$. $\mathcal{T}$ does not satisfy the representation, since otherwise from Theorem 8.10 would follow that $\mathcal{I}_\mathcal{T}(D,D)\subseteq D$.

Example 8.2: Let $\mathcal{T}$ be the $(L^*)^2 - L^*$ mapping defined by, for all x, $y\in L^*$,

$$\begin{aligned} \mathcal{T}(x,y) = (&\max(0,x_1+y_1-1),\\ &\min\left(1, x_2+y_2+\frac{1}{2}, 1-x_1+y_2, 1-y_1+x_2\right)\Bigg). \end{aligned}$$

Then $\mathcal{T}$ is a continuous, commutative, increasing mapping satisfying $\mathcal{T}(1_{L^*},y) = y$ and $\mathcal{T}((0,0),(0,0)) = (0,1/2)$.

We calculate, for x, y, and z in L^*

$$\begin{aligned} &pr_2\mathcal{T}(x,\mathcal{T}(y,z))\\ &\quad= \min\left(1, x_2+\min\left(1,y_2+z_2+\frac{1}{2}, 1-y_1+z_2,\right.\right.\\ &\qquad\qquad 1-z_1+y_2)+\frac{1}{2}, 1-x_1\\ &\qquad +\min\left(1,y_2+z_2+\frac{1}{2},\right.\\ &\qquad\qquad 1-y_1+z_2, 1-z_1+y_2)\\ &\qquad 1-\max(0,y_1+z_1-1)+x_2)\\ &\quad= \min(1,x_2+y_2+z_2+1, 1-y_1+x_2\\ &\qquad +z_2+\frac{1}{2}, 1-z_1+x_2+y_2+\frac{1}{2}, 1-x_1+y_2\\ &\qquad +z_2+\frac{1}{2}, 1-x_1+1-y_1+z_2, 1-x_1\\ &\qquad +1-z_1+y_2, 1-y_1+1-z_1+x_2) \end{aligned}$$

which is symmetrical in x and y. So $\mathcal{T}$ is associative, hence, $\mathcal{T}$ is an intuitionistic fuzzy t-norm.

Since $pr_1\mathcal{T}(x,x) < x_1$ for all $x \in L^*$ such that $x_1 \in]0,1[$, and since for all $x \in L^*$ such that $x_1 = 0$, $pr_2\mathcal{T}(x,x) = \min(1, 2x_2 + (1/2))$, $\mathcal{T}$ is Archimedean. Let $x = (0.1, 0.6)$, then $\mathcal{T}(x,x) = (0,1) = 0_{L^*}$. Hence, $\mathcal{T}$ is intuitionistic fuzzy nilpotent.

Now, we have, for x, y and z in L^*

$$\begin{aligned}&\mathcal{T}(x,y) \le_{L^*} z\\ &\quad\Leftrightarrow x_1 + y_1 - 1 \le z_1 \text{ and } x_2 + y_2 + \frac{1}{2} \ge z_2\\ &\qquad \text{and } 1 - x_1 + y_2 \ge z_2 \text{ and } 1 - y_1 + x_2 \ge z_2\\ &\quad\Leftrightarrow y_1 \le \min(1, z_1 + 1 - x_1, 1 - z_2 + x_2)\\ &\qquad \text{and } y_2 \ge \max\left(0, z_2 + x_1 - 1, z_2 - x_2 - \frac{1}{2}\right).\end{aligned}$$

Hence, $\mathcal{I}_\mathcal{T}(x,z) = (\min(1, z_1 + 1 - x_1, 1 - z_2 + x_2), \max(0, z_2 + x_1 - 1, z_2 - x_2 - (1/2)))$ and $\mathcal{T}$ satisfies the residuation principle, on the condition that $a = (\min(1, z_1+1-x_1, 1-z_2+x_2), \max(0, z_2+x_1-1, z_2-x_2-(1/2))) \in L^*$. Since $\min(1, z_1 + 1 - x_1, 1 - z_2 + x_2) + 0 \le 1 + 0$, $\min(1, z_1 + 1 - x_1, 1 - z_2 + x_2) + z_2 + x_1 - 1 \le z_1 + 1 - x_1 + z_2 + x_1 - 1 = z_1 + z_2 \le 1$ and $\min(1, z_1 + 1 - x_1, 1 - z_2 + x_2) + z_2 - x_2 - (1/2) \le 1 - z_2 + x_2 + z_2 - x_2 - (1/2) = (1/2) \le 1$, we have that $a \in L^*$ for all $x, z \in L^*$. Moreover, for all $x, z \in D$, we have $\mathcal{I}_\mathcal{T}(x,z) = (\min(1, z_1 + 1 - x_1), \max(0, x_1 - z_1)) \in D$.

So, $\mathcal{T}$ is an intuitionistic fuzzy t-norm satisfying all the conditions of Theorem 8.10 except $\mathcal{T}((0,0),(0,0)) = 0_{L^*}$.

IX. Representation of Intuitionistic Fuzzy T-Conorms

Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm satisfying the residuation principle, i.e. $\mathcal{T}(x,y) \le_{L^*} z \Leftrightarrow y \le_{L^*} \mathcal{I}_\mathcal{T}(x,z)$, for all x, y and z in L^*. Let $\mathcal{N}$ be the negator induced by $\mathcal{I}_\mathcal{T}$, i.e. $\mathcal{N}(x) = \mathcal{I}_\mathcal{T}(x, 0_{L^*})$ for all $x \in L^*$. Then, the dual intuitionistic fuzzy t-conorm $\mathcal{S}$ of $\mathcal{T}$ w.r.t. $\mathcal{N}$ is equal to $\mathcal{N} \circ \mathcal{S} \circ (\mathcal{N} \times \mathcal{N})$. Furthermore, we have $\mathcal{N}(\mathcal{S}(\mathcal{N}(x), \mathcal{N}(y))) \le_{L^*} z \Leftrightarrow y \le_{L^*} \mathcal{I}_\mathcal{T}(x,z)$. Assume that $\mathcal{N}$ is an involutive negator, then this is equivalent to $\mathcal{S}(\mathcal{N}(x), \mathcal{N}(y)) \ge_{L^*} \mathcal{N}(z) \Leftrightarrow y \le_{L^*} \mathcal{I}_\mathcal{T}(x,z)$, for all x, y, $z \in L^*$. So $\mathcal{S}(x,y) \ge_{L^*} z \Leftrightarrow \mathcal{N}(y) \le_{L^*} \mathcal{I}_\mathcal{T}(\mathcal{N}(x), \mathcal{N}(z)) \Leftrightarrow y \ge_{L^*} \mathcal{N}(\mathcal{I}_\mathcal{T}(\mathcal{N}(x), \mathcal{N}(z)))$, for all x, y, $z \in L^*$.

Define now the $(L^*)^2 - L^*$ mapping $\mathcal{I}_\mathcal{S}^c$ by $\mathcal{I}_\mathcal{S}^c(x,y) = \mathcal{N}(\mathcal{I}_\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y)))$, for all x, $y \in L^*$. Then $\mathcal{I}_\mathcal{S}^c$ is decreasing in its first and increasing in its second component, $\mathcal{I}_\mathcal{S}^c(x, 0_{L^*}) = 0_{L^*}$, $\mathcal{I}_\mathcal{S}^c(1_{L^*}, y) = 0_{L^*}$ and $\mathcal{I}_\mathcal{S}^c(0_{L^*}, 1_{L^*}) = 1_{L^*}$, for all $x, y \in L^*$. Moreover we have that $\mathcal{I}_\mathcal{S}^c(x, 1_{L^*}) = \mathcal{N}(\mathcal{I}_\mathcal{T}(\mathcal{N}(x), 0_{L^*})) = \mathcal{N}(\mathcal{N}(\mathcal{N}(x))) = \mathcal{N}(x)$. This gives rise to the definition of an intuitionistic fuzzy coimplicator, which is an extension of a fuzzy coimplicator [28].

Definition 9.1: An intuitionistic fuzzy coimplicator is an $(L^*)^2 - L^*$ mapping $\mathcal{I}^c$ satisfying the following conditions:

$$\begin{aligned}&\mathcal{I}^c(0_{L^*}, 0_{L^*}) = 0_{L^*} \quad \mathcal{I}^c(1_{L^*}, 0_{L^*}) = 0_{L^*}\\ &\mathcal{I}^c(1_{L^*}, 1_{L^*}) = 0_{L^*} \quad \mathcal{I}^c(0_{L^*}, 1_{L^*}) = 1_{L^*}\end{aligned}$$

$$\begin{aligned}&(\forall y \in L^*)\left(\forall (x,x') \in (L^*)^2\right)\\ &\quad (x \le_{L^*} x' \Rightarrow \mathcal{I}^c(x,y) \ge_{L^*} \mathcal{I}^c(x',y))\\ &(\forall x \in L^*)\left(\forall (y,y') \in (L^*)^2\right)\\ &\quad (y \le_{L^*} y' \Rightarrow \mathcal{I}^c(x,y) \le_{L^*} \mathcal{I}^c(x,y')).\end{aligned}$$

An intuitionistic fuzzy t-conorm $\mathcal{S}$ satisfies the residuation principle if and only if $\mathcal{S}(x,y) \ge_{L^*} z \Leftrightarrow y \ge_{L^*} \mathcal{I}_\mathcal{S}^c(x,z)$, for all x, y and z in L^*, where $\mathcal{I}_\mathcal{S}^c(x,z)$ is defined as $\mathcal{I}_\mathcal{S}^c(x,z) = \inf\{\gamma | \gamma \in L^* \text{ and } \mathcal{S}(x,\gamma) \ge_{L^*} z\}$.

Similarly, as in Section VII, the following theorems are proved.

Theorem 9.1: Let $\mathcal{S}$ be an intuitionistic fuzzy t-conorm. Then $\mathcal{S}$ satisfies the residuation principle if and only if

$$\inf_{z \in Z} \mathcal{S}(x,z) = \mathcal{S}\left(x, \inf_{z \in Z} z\right) \tag{14}$$

for any $x \in L^*$ and any subset Z of L^*.

Theorem 9.2: Let $\mathcal{S}$ be a t-representable intuitionistic fuzzy t-conorm. Then the partial mappings of $\mathcal{S}$ are intuitionistic fuzzy right-continuous if and only if $\mathcal{S}$ satisfies the residuation principle.

Before we can give the main result of this section, we first need some definitions and lemma's, which are similar to the ones given for intuitionistic fuzzy t-norms.

Definition 9.2: An intuitionistic fuzzy t-conorm $\mathcal{S}$ is called Archimedean if and only if, for all $x \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$, $\mathcal{S}(x,x) >_{L^*} x$.

Definition 9.3: An intuitionistic fuzzy t-conorm $\mathcal{S}$ is called nilpotent if and only if there exist $x, y \in L^* \setminus \{1_{L^*}\}$ such that $\mathcal{S}(x,y) = 1_{L^*}$.

An intuitionistic fuzzy t-conorm $\mathcal{S}$ is called intuitionistic fuzzy nilpotent if and only if there exist $x, y \in L^*$ such that x_2 and y_2 are nonzero and such that $\mathcal{S}(x,y) = 1_{L^*}$.

Lemma 9.3: Let $\mathcal{S}$ be an intuitionistic fuzzy t-conorm satisfying the residuation principle. Then, for any x, y, z in L^* such that $\mathcal{S}(x,y) = z$, there exists an $y' \in L^*$ such that $y' \le_{L^*} y$ and

$$\mathcal{S}(x,y') = z \quad \text{and} \quad y' = \mathcal{I}_\mathcal{S}^c(x,z). \tag{15}$$

Lemma 9.4: Let $\mathcal{S}$ be an intuitionistic fuzzy t-conorm satisfying the residuation principle, and x, y and y' arbitrary elements of L^*. Assume y and y' satisfy

$$\begin{aligned}\mathcal{S}(x,y) = z \quad &\text{and} \quad \mathcal{I}_\mathcal{S}^c(x,z) = y\\ \mathcal{S}(x,y') = z' \quad &\text{and} \quad \mathcal{I}_\mathcal{S}^c(x,z') = y'.\end{aligned}$$

Then $z \le_{L^*} z' \Leftrightarrow y \le_{L^*} y'$.

Lemma 9.5: Let $\mathcal{S}$ be an intuitionistic fuzzy t-conorm satisfying the residuation principle and $\mathcal{S}((0,0),(0,0)) = 1_{L^*}$. Then, for any x, $y \in L^*$, $pr_2\mathcal{S}(x,y)$ is independent of x_1 and y_1.

Lemma 9.6: Let $\mathcal{S}$ be a continuous Archimedean intuitionistic fuzzy nilpotent intuitionistic fuzzy t-conorm satisfying the residuation principle and $\mathcal{S}((0,0),(0,0)) = 1_{L^*}$. Then the mapping S, defined by $S(x_2, y_2) = 1 - pr_2\mathcal{S}((0, 1 -$

$x_2), (0, 1-y_2)) = 1 - pr_2\mathcal{S}((x_2, 1-x_2), (y_2, 1-y_2))$, for all x_2, $y_2 \in [0,1]$, is a continuous Archimedean nilpotent t-conorm.

Theorem 9.7: Let $\mathcal{S}$ be a continuous Archimedean intuitionistic fuzzy nilpotent intuitionistic fuzzy t-conorm satisfying the residuation principle and $\mathcal{S}((0,0),(0,0)) = 1_{L^*}$. Then there exists an increasing permutation φ of [0,1] such that, for any x, $y \in L^*$

$$pr_2\mathcal{S}(x,y) = 1 - \varphi^{-1}\left(\min\left(1, \varphi(1-x_2) + \varphi(1-y_2)\right)\right).$$

Lemma 9.8: Let $\mathcal{S}$ be an intuitionistic fuzzy t-conorm satisfying the residuation principle. Assume $\mathcal{I}^c_{\mathcal{S}}(D,D) \subseteq D$. Then for all $x \in D$ and $y_2 \in [0,1]$,

$$pr_1\mathcal{S}\left(x, (y_1, 0)\right) = pr_1\mathcal{S}\left(x, (y_1, 1-y_1)\right).$$

Lemma 9.9: Let $\mathcal{S}$ be an intuitionistic fuzzy t-conorm satisfying the residuation principle and $\mathcal{S}((0,0),(0,0)) = 1_{L^*}$. Then for all $y_1, x_2 \in [0,1]$,

$$pr_1\mathcal{S}\left((0, x_2), (y_1, 0)\right) = pr_1\mathcal{S}\left((1-x_2, x_2), (y_1, 0)\right).$$

Theorem 9.10: Let $\mathcal{S}$ be a continuous Archimedean intuitionistic fuzzy nilpotent intuitionistic fuzzy t-conorm satisfying the residuation principle, $\mathcal{I}^c_{\mathcal{S}}(D,D) \subseteq D$ and $\mathcal{S}((0,0),(0,0)) = 1_{L^*}$. Then for all x, $y \in L^*$

$$pr_1\mathcal{S}(x,y) = \varphi^{-1}\left(\min\left(1, \varphi(1-x_2) + \varphi(y_1), \varphi(1-y_2) + \varphi(x_1)\right)\right)$$

where φ is the permutation from Theorem 9.7.

Define the $(L^*)^2 - L^*$ mapping $\mathcal{S}_W$ as $\mathcal{S}_W(x,y) = (\min(1, 1-x_2+y_1, 1-y_2+x_1), 1-\min(1, 1-x_2+1-y_2))$, for all x, $y \in L^*$. Then $\mathcal{S}_W$ is the dual intuitionistic fuzzy t-conorm of $\mathcal{T}_W$ w.r.t. the standard negator $\mathcal{N}_s$. The residual coimplicator $\mathcal{I}^c_{\mathcal{S}_W}$ of $\mathcal{S}_W$ is given by $\mathcal{I}^c_{\mathcal{S}_W}(x,z) = (\max(0, z_1+x_2-1), \min(1, z_2+1-x_2, x_1+1-z_1))$. It holds that $\mathcal{I}^c_{\mathcal{S}_W} = \mathcal{N}_s \circ \mathcal{I}_{\mathcal{T}_W} \circ (\mathcal{N}_s \times \mathcal{N}_s)$.

Theorem 9.11: Let $\mathcal{S}$ be an $(L^*)^2 - L^*$ mapping. Assume there exists a continuous increasing permutation Φ of L^* with increasing inverse such that $\mathcal{S} = \Phi^{-1} \circ \mathcal{S}_W \circ (\Phi \times \Phi)$. Then $\mathcal{S}$ is a continuous, Archimedean, intuitionistic fuzzy nilpotent intuitionistic fuzzy t-conorm satisfying the residuation principle, $\mathcal{I}^c_{\mathcal{S}}(D,D) \subseteq D$ and $\mathcal{S}((0,0),(0,0)) = 1_{L^*}$.

Now, we can give the main result.

Theorem 9.12: Let $\mathcal{S}$ be an $(L^*)^2 - L^*$ mapping. Then the following are equivalent.

i) $\mathcal{S}$ is a continuous, Archimedean, intuitionistic fuzzy nilpotent intuitionistic fuzzy t-conorm satisfying the residuation principle, $\mathcal{I}^c_{\mathcal{S}}(D,D) \subseteq D$ and $\mathcal{S}((0,0),(0,0)) = 1_{L^*}$.

ii) There exists a continuous increasing permutation φ of [0, 1] such that, for all $x, y \in L^*$

$$\mathcal{S}(x,y) = \Big(\varphi^{-1}\left(\min\left(1, \varphi(1-x_2)+\varphi(y_1), \varphi(1-y_2)+\varphi(x_1)\right)\right) \; 1-\varphi^{-1}\left(\min\left(1, \varphi(1-x_2)+\varphi(1-y_2)\right)\right)\Big). \quad (16)$$

iii) There exists a continuous increasing permutation Φ of L^* with increasing inverse such that $\mathcal{S} = \Phi^{-1} \circ \mathcal{S}_W \circ (\Phi \times \Phi)$.

For any intuitionistic fuzzy coimplicator $\mathcal{I}^c$, we call the $L^* - L^*$ mapping $\mathcal{N}$ defined by $\mathcal{N}(x) = \mathcal{I}^c(x, 1_{L^*})$ the negator induced by $\mathcal{I}^c$. This is indeed an intuitionistic fuzzy negator since $\mathcal{I}^c$ is decreasing in its first component, $\mathcal{I}^c(0_{L^*}, 1_{L^*}) = 1_{L^*}$ and $\mathcal{I}^c(1_{L^*}, 1_{L^*}) = 0_{L^*}$.

Note that $\mathcal{I}^c_{\mathcal{S}}(x,z) = (\varphi^{-1}(\max(0, \varphi(z_1)-\varphi(1-x_2))), 1-\varphi^{-1}(1-\min(1, 1-\varphi(1-z_2)+\varphi(1-x_2), \varphi(x_1)+1-\varphi(z_1))))$. Moreover $\mathcal{N}(x) = \mathcal{I}^c_{\mathcal{S}}(x, 1_{L^*}) = (\varphi^{-1}(1-\varphi(1-x_2)), 1-\varphi^{-1}(1-\varphi(x_1)))$. So $\mathcal{N}(x) = (N(1-x_2), 1-N(x_1))$, with $N = \varphi^{-1} \circ N_s \circ \varphi$.

Theorem 9.13: Let $\mathcal{S}$ be an intuitionistic fuzzy t-conorm and assume there exists a continuous increasing permutation φ of [0, 1] such that (16) holds. Let $\mathcal{N}$ be the negator induced by $\mathcal{I}^c_{\mathcal{S}}$. Then the dual intuitionistic fuzzy t-norm $\mathcal{T}$ of $\mathcal{S}$ w.r.t. $\mathcal{N}$ satisfies (13) for the same φ.

Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm and assume there exists a continuous increasing permutation φ of [0, 1] such that (13) holds. Let $\mathcal{N}$ be the negator induced by $\mathcal{I}_{\mathcal{T}}$. Then the dual intuitionistic fuzzy t-conorm $\mathcal{S}$ of $\mathcal{T}$ w.r.t. $\mathcal{N}$ satisfies (16) for the same φ.

Proof: Let $\mathcal{S}$ be an intuitionistic fuzzy t-conorm and assume there exists a continuous increasing permutation φ of [0, 1] such that (16) holds. Then from the above follows that the negator $\mathcal{N}$ induced by $\mathcal{I}^c_{\mathcal{S}}$ is given by $\mathcal{N}(x) = (\varphi^{-1}(1-\varphi(1-x_2)), 1-\varphi^{-1}(1-\varphi(x_1)))$. Then

$$\begin{aligned}\mathcal{T}(x,y) &= \mathcal{N}\left(\mathcal{S}\left(\mathcal{N}(x), \mathcal{N}(y)\right)\right)\\ &= \Big(\varphi^{-1}\left(1-\varphi\left(1-pr_2\mathcal{S}\left(\mathcal{N}(x), \mathcal{N}(y)\right)\right)\right)\\ &\quad 1-\varphi^{-1}\left(1-\varphi\left(pr_1\mathcal{S}\left(\mathcal{N}(x), \mathcal{N}(y)\right)\right)\right)\Big)\\ &= \Big(\varphi^{-1}\left(1-\min\left(1, 1-\varphi(x_1)+1-\varphi(y_1)\right)\right)\\ &\quad 1-\varphi^{-1}\left(1-\min\left(1, 1-\varphi(x_1)+1-\varphi(1-y_2), 1-\varphi(y_1)+1-\varphi(1-x_2)\right)\right)\Big)\\ &= \Big(\varphi^{-1}\left(\max\left(0, \varphi(x_1)+\varphi(y_1)-1\right)\right)\\ &\quad 1-\varphi^{-1}\left(\max\left(0, \varphi(x_1)+\varphi(1-y_2)-1, \varphi(y_1)+\varphi(1-x_2)-1\right)\right)\Big).\end{aligned}$$

Hence, $\mathcal{T}$ satisfies (13).

The second part is proved analogously. ■

X. Conclusion

We have extended the notion of triangular norm and conorm to intuitionistic fuzzy set theory, and also the notion of Archimedean property and nilpotency. In fuzzy set theory a t-norm satisfies the residuation principle if and only if it is left-continuous. We have shown that for intuitionistic fuzzy t-norms intuitionistic fuzzy left-continuity is a necessary but not sufficient condition for the residuation principle to hold. We have introduced the intuitionistic fuzzy Łukasiewicz t-norm $\mathcal{T}_W$ and established the necessary and sufficient conditions for an intuitionistic fuzzy t-norm $\mathcal{T}$ such that there exists a permutation Φ of L^* such that $\mathcal{T}$ is the Φ-transform of $\mathcal{T}_W$. Similarly we have found a representation theorem for a subclass of intuitionistic fuzzy t-conorms. In order to prove these representation theorems we had to establish a representation for involutive negators and continuous increasing permutations. Several distances on L^* have been defined in the literature (see, e.g., [19] and [26]). We have shown that these distances are topologically equivalent, so continuity in L^* can be investigated using either of these distances. In another paper [29], we give a representation for intuitionistic fuzzy implicators based on the representations given here.

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Glad Deschrijver received the B.Sc. and M.Sc. degrees in mathematics from Ghent University, Ghent, Belgium, in 1997 and 1999, respectively. He is currently working toward the Ph.D. degree at the same university.

His research interests include the mathematics of different models of imprecision, including interval-valued fuzzy sets, intuitionistic fuzzy sets, and L-fuzzy sets. He is currently doing research on intuitionistic fuzzy t-norms and related operators.

Chris Cornelis received the B.Sc. and M.Sc. degrees in computer science from Ghent University, Belgium, in 1998 and 2000, respectively. He is currently working toward the Ph.D. degree, sponsored by a grant from the National Science Foundation-Flanders, at the same university.

His research interests include mathematical models of imprecision, including interval-valued fuzzy sets, intuitionistic fuzzy sets, L-fuzzy sets, rough sets, and various combinations of these. He applied these models in approximate reasoning, possibility theory, and data mining.

Etienne E. Kerre was born in Zele, Belgium, in 1945. He received the M.Sc. and Ph.D. degrees in mathematics Ghent University, Ghent University, Ghent, Belgium, in 1967 and 1970, respectively.

Since 1984, he has been a Lector, and since 1991, a Full Professor at Ghent University. He is a referee for more than 30 international scientific journals, and also a Member of the Editorial Board of international journals and conferences on fuzzy set theory. He was an Honorary Chairman at various international conferences. In 1976, he founded the Fuzziness and Uncertainty Modeling Research Unit (FUM) and, since then, his research has been focused on the modeling of fuzziness and uncertainty, and has resulted in a great number of contributions in fuzzy set theory and its various generalizations, and in evidence theory. The theories of fuzzy relational calculus and of fuzzy mathematical structures owe a very great deal to him. Over the years, he has also been a promotor of 16 Ph.D. degrees on fuzzy set theory. His current research interests include fuzzy and intuitionistic fuzzy relations, fuzzy topology, and fuzzy image processing. He has authored or coauthored 11 books, and more than 100 papers appeared in international refereed journals.