Triangle Algebras: towards an Axiomatization of Interval-Valued Residuated Lattices

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Abstract. In this paper, we present triangle algebras: residuated lattices equipped with two modal, or approximation, operators and with a third angular point u, different from 0 (false) and 1 (true), intuitively denoting ignorance about a formula's truth value. We prove that these constructs, which bear a close relationship to several other algebraic structures including rough approximation spaces, provide an equational representation of interval-valued residuated lattices, which are triangularizations of residuated lattices; as an important case in point, we consider $\mathcal{L}^I$, the lattice of closed intervals of $[0,1]$. As we will argue, the representation by triangle algebras serves as a crucial stepping stone to the construction of formal interval-valued fuzzy logics, and in particular to the axiomatic formalization of residuated t-norm based logics on $\mathcal{L}^I$, in a similar way as was done for formal fuzzy logics on the unit interval.

1 Introduction and Preliminaries

Formal fuzzy logics (also: fuzzy logics in the narrow sense) are generalizations of classical logic that allow us to reason gradually. Indeed, in the scope of these logics, formulas can be assigned not only 0 and 1 as truth values, but also elements of [0,1], or, more generally, of a bounded lattice $\mathcal{L}$. The partial ordering of $\mathcal{L}$ then serves to compare the truth values of formulas which can be true to some extent. The best-known examples of formal fuzzy logics are probably Monoidal T-norm based Logic (MTL, Esteva and Godo [11]), Basic Logic (BL, Hájek [14]), Gödel logic (G, [13]) and Łukasiewicz logic (Ł, [15]). For all of these logics, which are fully described in terms of axioms, with the modus ponens as deduction rule, soundness and completeness with respect to a corresponding variety[1] can be proved. For instance, a formula can be deduced in MTL iff it is true (i.e., has truth value 1) in every prelinear residuated lattice; recall that a residuated lattice is a structure $\mathcal{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ in which $\sqcap, \sqcup, *$ and $\Rightarrow$ are binary operators on L and

- $(L, \sqcap, \sqcup)$ is a bounded lattice with 0 as smallest and 1 as greatest element,

[1] Recall that a class $\mathcal{K}$ of structures is a variety [14] if there is a set T of identities such that $\mathcal{K}$ is the class of structures in which all identities from T are true.

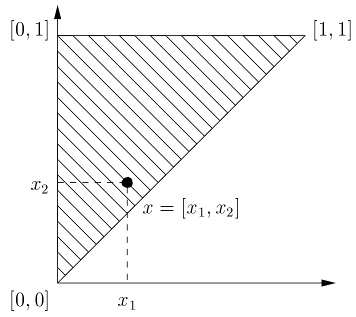

Fig. 1. The lattice $\mathcal{L}^I$

- $*$ is commutative and associative, with 1 as neutral element, and
- $x * y \leq z$ iff $x \leq (y \Rightarrow z)$ for all x, y and z in L (residuation principle),

and that prelinearity means that $(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$ for all x and y in L. A prelinear residuated lattice is called an MTL-algebra. The other logics emerge by adding axioms to MTL, and are sound and complete w.r.t. subvarieties of MTL-algebras. For a comprehensive overview of the state-of-the-art on formal fuzzy logics, we refer to [12].

Research on formal fuzzy logics has centered on prelinear residuated structures; indeed, all of the above-mentioned logics presuppose prelinearity. However, while the property holds in every residuated lattice $([0,1], \min, \max, *, \Rightarrow, 0, 1)$[2], it is not necessarily preserved for closed intervals of a bounded lattice $\mathcal{L}$; for example, it was shown in [6] that no MTL-algebra exists on the lattice $\mathcal{L}^I = (L^I, \bigsqcap, \bigsqcup)$, shown graphically in Figure 1 and defined by

$$\begin{array}{l} L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0,1]^2 \text{ and } x_1 \leq x_2\} \\ \text{and} \\ [x_1, x_2] \bigsqcap [y_1, y_2] = [\min(x_1, y_1), \min(x_2, y_2)] \\ [x_1, x_2] \bigsqcup [y_1, y_2] = [\max(x_1, y_1), \max(x_2, y_2)], \end{array}$$

for all $[x_1, x_2]$ and $[y_1, y_2]$ in L^I; and whose partial ordering $\leq_{L^I}$ is given by componentwise extension of $\leq$,

$$[x_1, x_2] \leq_{L^I} [y_1, y_2] \iff x_1 \leq y_1 \text{ and } x_2 \leq y_2.$$

This is not to say that such structures are of no significance for logical purposes. Indeed, note that elements drawn from $\mathcal{L}^I$, or more generally from the lattice of closed intervals of a bounded lattice $\mathcal{L}$, which in this paper we shall call triangularizations of $\mathcal{L}$, carry an attractive and straightforward semantical interpretation as partial, or incomplete, truth values, i.e. they exhibit a lack of

[2] Recall that $([0,1], \min, \max, *, \Rightarrow, 0, 1)$ is a residuated lattice iff $*$ is a left-continuous t-norm on $[0,1]$.

knowledge about a formula's exact truth value; the wider the interval, the greater the uncertainty. Note that the angular point $[0,1]$ in Figure 1 corresponds to "ignorance", or total uncertainty about the exact truth value. This interpretation, together with the relative efficiency of operations defined on them, accounts for the widespread adoption and application of interval-valued truth degrees in knowledge-based systems (see e.g. [16,17]). Moreover, residuated lattices can be constructed on top of triangularizations quite easily, and as extensive research in the context of $\mathcal{L}^I$ (see e.g. [5]) has pointed out, some of them rival their counterparts on [0,1] for the properties they satisfy.

The goal of this paper is to characterize interval-valued residuated lattices (which are residuated lattices on triangularizations) as a variety, i.e. by a set of identities that capture their triangular structure (as seen also in Figure 1). This is not only interesting from a purely mathematical stance, it also paves the way for the development of formal interval-valued fuzzy logics, since identities are much more readily axiomatizable than the structural description as triangularizations. A natural and convenient way to obtain this algebraic characterization is the introduction of modal, or approximation, operators. Such operators have been studied from various angles [1,3,14,19] and serve well to describe the incompleteness facet of interval-valued residuated lattices. They give rise to the introduction of triangle algebras in Section 2. In Section 3 we review some related algebraic structures. In Section 4, we prove that every triangle algebra uniquely determines an interval-valued residuated lattice, and vice versa. To illustrate the relevance of these concepts, we relate them to existing work about residuated t-norms (Section 5) on $\mathcal{L}^I$. Finally, Section 6 offers a conclusion and discusses future work.

2 Triangle Algebras

As mentioned in the previous section, we want to construct an algebra that captures the triangular structure of interval-valued residuated lattices (Definition 3) by a set of appropriate conditions. To this end, we extend the definition of a residuated lattice with a new constant u ("uncertainty") and two new unary connectives ν ("necessity") and μ ("possibility"); intuitively, the elements of a triangle algebra may be thought of as intervals; the formal link with interval-valued residuated lattices will be established in Section 4.

Definition 1. A triangle algebra is a structure $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, in which $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice, and in which the following 17 conditions hold ($x \Leftrightarrow y$ is a shorthand notation for $(x \Rightarrow y) \sqcap (y \Rightarrow x)$):

$T.1\ \nu x \leq x$	$T.1'\ x \leq \mu x$
$T.2\ \nu x \leq \nu\nu x$	$T.2'\ \mu\mu x \leq \mu x$
$T.3\ \nu(x \sqcap y) = \nu x \sqcap \nu y$	$T.3'\ \mu(x \sqcap y) = \mu x \sqcap \mu y$
$T.4\ \nu(x \sqcup y) = \nu x \sqcup \nu y$	$T.4'\ \mu(x \sqcup y) = \mu x \sqcup \mu y$
$T.5\ \nu 1 = 1$	$T.5'\ \mu 0 = 0$
$T.6\ \nu u = 0$	$T.6'\ \mu u = 1$
$T.7\ \nu\mu x = \mu x$	$T.7'\ \mu\nu x = \nu x$

$T.8 \quad \nu(x \Rightarrow y) \leq \nu x \Rightarrow \nu y$
$T.9 \quad (\nu x \Leftrightarrow \nu y) * (\mu x \Leftrightarrow \mu y) \leq (x \Leftrightarrow y)$
$T.10 \; \nu x \Rightarrow \nu y \leq \nu(\nu x \Rightarrow \nu y)$

Remark 1. Suppose $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice such that the negation $\neg$, defined by $\neg x = x \Rightarrow 0$, is involutive (i.e., $\neg\neg x = x$ for every x in A). If there exists an element u in A such that $\neg u = u$, if ν is a unary operator on A that satisfies (T.1–T.6, T.8, T.10), and if $(\nu x \Leftrightarrow \nu y) * (\nu\neg x \Leftrightarrow \nu\neg y) \leq (x \Leftrightarrow y)$, then $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is a triangle algebra if we define $\mu x = \neg\nu\neg x$. In general, however, there need not be a link between ν and μ.

Denote the set of exact elements of a triangle algebra $\mathcal{A}$ by $E(\mathcal{A}) = \{x \in A \mid \nu x = x\}$. By the fact that the following statements are equivalent for all x in A:

1. $x = \nu y$ for some y in A
2. $x = \mu y$ for some y in A
3. $x = \nu x$
4. $x = \mu x$
5. $\nu x = \mu x$

it is clear that $E(\mathcal{A})$ is the direct image of A under ν, as well as under μ. Moreover, this set is invariant under ν and μ, and contains 0 and 1, but not u. It is closed under $\sqcap$, $\sqcup$, $*$ and $\Rightarrow$, and hence it also holds that $\mathcal{E}(\mathcal{A}) = (E(\mathcal{A}), \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice.

3 Connections to Other Algebraic Structures

The idea of introducing modal-like operators in residuated lattices and other algebraic structures has also been adopted by other authors, for several purposes.

- Belohlávek and Vychodil [1] defined a so-called "truth stresser" ν for a residuated lattice $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ as a unary operator on L that satisfies T.1, T.5 and T.8. They used it to model the (truth function of a) unary connective "very true".
- Ono [19] defined modal residuated lattices as structures $(L, \sqcap, \sqcup, *, \Rightarrow, \nu, 0, 1)$, in which $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice and ν a unary operator on L that satisfies T.1, T.2, T.5, and, for all x and y in L, $\nu(x \sqcap y) \leq \nu x$ and $\nu x * \nu y \leq \nu(x * y)$. We can prove, by the residuation principle, that the latter two properties are equivalent to T.8. Hence, in a modal residuated lattice, ν is a truth stresser additionally satisfying T.2; and if $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is a triangle algebra, then $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, 0, 1)$ is a modal residuated lattice.
- A Hájek [14] truth stresser for a residuated lattice $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a unary operator ν on L that satisfies T.1, T.2, T.5, T.8, $\nu(x \sqcup y) \leq \nu x \sqcup \nu y$ (which is in this case equivalent to T.4) and $\nu x \sqcup \neg\nu x = 1$ (weakened law of excluded middle, WLEM) for every x and y in L. Hence, $(L, \sqcap, \sqcup, *, \Rightarrow, \nu, 0, 1)$

is a modal residuated lattice in which T.4 and WLEM are satisfied. Hájek used this truth stresser to establish a faithful imbedding of Boolean logic into his $BL_\triangle$.

Triangle algebras do not maintain WLEM as, in many cases, it would imply that $\nu x = 0$ whenever $x \neq 1$. This is not compatible with our interpretation of "necessity of an interval": for example (on $\mathcal{L}^I$), the necessity of $[0.9, 0.9]$ should be greater than the necessity of $[0.2, 0.5]$, it should not be the case that both are $[0, 0]$. However, we do impose several other conditions. T.1'–T.5' are conditions for possibility, dual to T.1–T.5 (in general, we do not require dependency of μ on ν; an example in which this holds is considered in Remark 1). The conditions T.6 and T.6' express the complete lack of knowledge about u: its necessity is 0, but its possibility is 1; T.7 and T.7' are known in modal logics as the S5-principles [18]. Condition T.9 implies that an element of a triangle algebra is completely defined by its necessity and possibility. Indeed: if $\nu x = \nu y$ and $\mu x = \mu y$, then $\nu x \Leftrightarrow \nu y = 1$ and $\mu x \Leftrightarrow \mu y = 1$, so $1 = 1 * 1 \leq x \Leftrightarrow y$, which implies $x = y$. Finally T.10 is a technical condition needed to ensure that triangle algebras correspond to interval-valued residuated lattices.

We adopted the notations ν and μ from Cattaneo and Ciucci [3], who defined these operators on so-called weak Brouwer de Morgan lattices (wBD lattices). A wBD lattice $(L, \sqcap, \sqcup, ', ^\sim, 0, 1)$ is a bounded distributive lattice $(L, \sqcap, \sqcup)$ equipped with two complementations:

- a de Morgan complementation $'$, which is defined as an involutive unary operator on L that satisfies[3] $(x \sqcup y)' = x' \sqcap y'$, for all x and y in L, and
- a weak Brouwer complementation $\sim$, which is defined as a unary operator satisfying $x \leq x^{\sim\sim}$ and $(x \sqcup y)^\sim = x^\sim \sqcap y^\sim$ for all x and y in L,

for which $x^{\sim\prime} = x^{\sim\sim}$ (interconnection rule).
They defined νx as $x'^\sim$ and μx as $x^{\sim\prime}$. In this structure, T.1, T.1', T.2, T.2', T.3, T.4', T.5, T.5', T.7 and T.7' are always fulfilled, as well as $\mu x = (\nu x')'$. Note that T.3' and T.4 are not always satisfied, because $(x \sqcap y)^\sim$ is not necessarily equal to $x^\sim \sqcup y^\sim$.
Some triangle algebras can be seen as wBD lattices:

Proposition 1. If $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is a distributive triangle algebra, if $'$ is a de Morgan complementation on A such that $\mu x = (\nu x')'$ and if we define $^\sim$ by $x^\sim = (\mu x)'$, then $(A, \sqcap, \sqcup, ', ^\sim, 0, 1)$ is a wBD lattice.

Finally, it can be seen that a triangle algebra $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ induces a rough approximation space $\mathcal{R} = (A, E(\mathcal{A}), \nu, \mu)$ (in the sense of Cattaneo [2]) in which

- A is the set of approximable elements,
- $E(\mathcal{A})$ is the set of exact or 'definable' elements,

[3] In this case, also $(x \sqcap y)' = x' \sqcup y'$ holds for every x and y in L.

- ν: $A \to E(\mathcal{A})$ is the inner approximation map, satisfying $(\forall x \in E(\mathcal{A}))(\forall y \in A)(x \leq y \text{ iff } x \leq \nu y)$,
- μ: $A \to E(\mathcal{A})$ is the outer approximation map, satisfying $(\forall x \in A)(\forall y \in E(\mathcal{A}))(x \leq y \text{ iff } \mu x \leq y)$,

and in which for any element x in A, its rough approximation is defined by $(\nu x, \mu x)$. In this case, T.9 ensures that no two different elements have the same rough approximation.

4 Connection with Interval-Valued Residuated Lattices

Definition 2. Given a lattice $\mathcal{L} = (L, \sqcap, \sqcup)$, its triangularization is the structure $\mathcal{T}(\mathcal{L}) = (T(\mathcal{L}), \bigsqcap, \bigsqcup)$ defined by

- $T(\mathcal{L}) = \{[x_1, x_2] \mid (x_1, x_2) \in L^2 \text{ and } x_1 \leq x_2\}$

and

- $[x_1, x_2] \bigsqcap [y_1, y_2] = [x_1 \sqcap y_1, x_2 \sqcap y_2]$
- $[x_1, x_2] \bigsqcup [y_1, y_2] = [x_1 \sqcup y_1, x_2 \sqcup y_2]$

for all $[x_1, x_2]$ and $[y_1, y_2]$ in $T(\mathcal{L})$. The set $D = \{[x, x] \mid x \in L\}$ is called the diagonal of $\mathcal{T}(\mathcal{L})$, and can be seen as a 'copy' of L inside $T(\mathcal{L})$. As an example, note that $\mathcal{L}^I$ is the triangularization of $([0, 1], \min, \max)$.

It is easy to verify that $\mathcal{T}(\mathcal{L})$ is again a lattice. If $\mathcal{L}$ contains a smallest element 0 (resp. a greatest element 1), then $T(\mathcal{L})$ has $[0, 0]$ as smallest element (resp. $[1, 1]$ as greatest element). Moreover, if $\mathcal{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice, then it is always possible to construct a residuated lattice on $\mathcal{T}(\mathcal{L})$; in particular, if we define

$$[x_1, x_2] \odot [y_1, y_2] = [x_1 * y_1, x_2 * y_2] \tag{1}$$

$$[x_1, x_2] \Rightarrow_\odot [y_1, y_2] = [(x_1 \Rightarrow y_1) \sqcap (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2] \tag{2}$$

then the structure $(T(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0, 0], [1, 1])$ is a residuated lattice. It is not the only possible way of defining residuated lattices on $T(\mathcal{L})$; Section 5 investigates other possibilities, on L^I. In general, we consider the following construct:

Definition 3. An interval-valued residuated lattice is a residuated lattice $(T(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0, 0], [1, 1])$ on the triangularization $\mathcal{T}(\mathcal{L})$ of a bounded lattice $\mathcal{L}$, in which D is closed under $\odot$ and $\Rightarrow_\odot$, i.e., $[x_1, x_1] \odot [y_1, y_1] \in D$ and $[x_1, x_1] \Rightarrow_\odot [y_1, y_1] \in D$ for x_1, y_1 in L.

Remark 2. Note that, under our assumptions, $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$, with $*$ and $\Rightarrow$ the restrictions of $\odot$ and $\Rightarrow_\odot$ to D, is always a residuated lattice. Our definition of interval-valued residuated lattice excludes those cases, in which $\odot$ and $\Rightarrow_\odot$ do not extend corresponding connectives on $\mathcal{L}$.

Proposition 2. If $(T(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, \nu, \mu, [0,0], [0,1], [1,1])$ is a triangle algebra on a triangularization $(T(\mathcal{L}), \bigsqcap, \bigsqcup)$ of a bounded lattice, then $\nu[x_1, x_2] = [x_1, x_1]$ and $\mu[x_1, x_2] = [x_2, x_2]$ for every $[x_1, x_2]$ in $T(\mathcal{L})$.

Proposition 2 makes clear the intended meaning of 'necessity' and 'possibility' of an interval: if $x = [x_1, x_2]$ is the incompletely specified truth value of a formula, then $\nu x = [x_1, x_1]$ and $\mu x = [x_2, x_2]$ represent the minimum, resp. maximum, exact truth value that emerges when the uncertainty is resolved. The next important theorem establishes triangle algebras as the equational representation of interval-valued residuated lattices.

Theorem 1.
If $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is a triangle algebra, then $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is isomorphic to an interval-valued residuated lattice.
Conversely, if $(A, \bigsqcap, \bigsqcup, *, \Rightarrow, [0,0], [1,1])$ is an interval-valued residuated lattice and ν and μ are defined by $\nu[x_1, x_2] = [x_1, x_1]$ and $\mu[x_1, x_2] = [x_2, x_2]$, then $(A, \bigsqcap, \bigsqcup, *, \Rightarrow, \nu, \mu, [0,0], [0,1], [1,1])$ is a triangle algebra.

We give a sketch of the proof.
For any triangle algebra $\mathcal{A}$, we can define the mapping ϕ: $A \longrightarrow T(\mathcal{E}(\mathcal{A}))$ as $\phi(x) = [\nu x, \mu x]$. This mapping is an injection because of condition T.9. Because of conditions T.3,T.3',T.4 and T.4' it is a homomorphism from $(A, \sqcap, \sqcup)$ to $(T(\mathcal{E}(\mathcal{A})), \bigsqcap, \bigsqcup)$: $\phi(x \sqcap y) = [\nu(x \sqcap y), \mu(x \sqcap y)] = [\nu x \sqcap \nu y, \mu x \sqcap \mu y] = [\nu x, \mu x] \bigsqcap [\nu y, \mu y] = \phi(x) \bigsqcap \phi(y)$ and analogously $\phi(x \sqcup y) = \phi(x) \bigsqcup \phi(y)$. It turns out that ϕ is also a surjection: for every $[x, y]$ in $T(\mathcal{E}(\mathcal{A}))$, $\nu x = x = \mu x$ and $\nu y = y = \mu y$; so

$$\begin{aligned}
[x, y] &= [\nu x, \mu x] \bigsqcup [0, \mu y] \\
&= [\nu x, \mu x] \bigsqcup ([0,1] \bigsqcap [\nu y, \mu y]) \\
&= \phi(x) \bigsqcup (\phi(u) \bigsqcap \phi(y)) \\
&= \phi(x \sqcup (u \sqcap y)).
\end{aligned}$$

We define on $T(\mathcal{E}(\mathcal{A}))$ the binary operation $\odot$ as $\phi(x) \odot \phi(y) = \phi(x * y)$. It follows immediately from this definition that ϕ is a homomorphism from $(A, *)$ to $(T(\mathcal{E}(\mathcal{A})), \odot)$. Since $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice, we know that $x \Rightarrow y = \sup\{z \in A \mid z * x \leq y\}$, so if we define $x \Rightarrow_\odot y$ as $\sup\{z \in T(\mathcal{E}(\mathcal{A})) \mid z \odot x \leq y\}$, ϕ is also a homomorphism from $(A, \Rightarrow)$ to $(T(\mathcal{E}(\mathcal{A})), \Rightarrow_\odot)$. Thus the structure $(T(\mathcal{E}(\mathcal{A})), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ is a residuated lattice, isomorphic to $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$. This means that every triangle algebra $\mathcal{A}$ has the structure of the set of intervals of a residuated lattice (its exact elements).

For the second part of the proof, assume that $(T(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ is an interval-valued residuated lattice and that ν and μ are defined by $\nu[x_1, x_2] = [x_1, x_1]$ and $\mu[x_1, x_2] = [x_2, x_2]$. Then it can be proven that $(T(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, \nu, \mu, [0,0], [0,1], [1,1])$ satisfies T.1-T.10 and T.1'-T.7'.

5 The case of $\mathcal{L}^I$

By extension of the corresponding notion on $[0, 1]$, t-norms on a bounded lattice $(L, \sqcap, \sqcup, 0, 1)$ are defined as increasing, associative, commutative mappings $\mathcal{T}$ that satisfy $\mathcal{T}(1, x) = x$ for x in L. Recall that such a t-norm $\mathcal{T}$ is called residuated if it induces a residuated lattice on L, that is, if $(L, \sqcap, \sqcup, \mathcal{T}, \mathcal{I}_{\mathcal{T}}, 0, 1)$ is a residuated lattice with $\mathcal{I}_{\mathcal{T}}(x, y) = \sup\{z \mid z \in L \text{ and } \mathcal{T}(x, z) \le y\}$. As mentioned in the introduction, a t-norm on [0,1] is residuated iff it is left-continuous; this property however does not extend to $\mathcal{L}^I$ [8]. While a general characterization of residuated t-norms on $\mathcal{L}^I$ has not yet been found, it was shown in [7] that if T induces a residuated lattice on $[0, 1]$, then for each $\alpha \in [0, 1]$, $\mathcal{T}_{T,\alpha}$ defined by, for $x = [x_1, x_2]$ and $y = [y_1, y_2]$ in L^I,

$$\mathcal{T}_{T,\alpha}(x, y) = [T(x_1, y_1), \max(T(\alpha, T(x_2, y_2)), T(x_1, y_2), T(x_2, y_1))], \qquad (3)$$

induces a residuated lattice on L^I. As the diagonal of $\mathcal{L}^I$ is closed under $\mathcal{T}_{T,\alpha}$ and $\mathcal{I}_{\mathcal{T}_{T,\alpha}}$, Theorem 1 implies that $(L^I, \sqcap, \sqcup, \mathcal{T}_{T,\alpha}, \mathcal{I}_{\mathcal{T}_{T,\alpha}}, \nu, \mu, [0, 0], [0, 1], [1, 1])$ is a triangle algebra.

Two important values of α can be distinguished in Formula (3):

- If $\alpha = 1$, we obtain t-representable t-norms on $\mathcal{L}^I$: $\mathcal{T}_{T,1}(x, y) = [T(x_1, y_1), T(x_2, y_2)]$, which can be seen as the straightforward (and most commonly used) extension of T to $\mathcal{L}^I$.
- If $\alpha = 0$, we obtain pseudo t-representable t-norms on $\mathcal{L}^I$: $\mathcal{T}_{T,0}(x, y) = [T(x_1, y_1), \max(T(x_1, y_2), T(x_2, y_1))]$. These t-norms are inherently more complex than their t-representable counterparts, but as we shall see below satisfy more relevant properties.

Just like on the unit interval, we can study particular subclasses of residuated t-norms on $\mathcal{L}^I$. First recall that a t-norm $\mathcal{T}$ on $(L, \sqcap, \sqcup, 0, 1)$ is called divisible if $\mathcal{T}(x, \mathcal{I}_{\mathcal{T}}(x, y)) = x \sqcap y$ and involutive if $\mathcal{I}_{\mathcal{T}}(\mathcal{I}_{\mathcal{T}}(x, 0), 0) = x$ for x, y in L, that a BL-algebra is a divisible residuated lattice, and that an MV-algebra is an involutive BL-algebra. On the unit interval, a t-norm induces a BL-algebra iff it is continuous, and an MV-algebra iff it is isomorphic to the Łukasiewicz t-norm T_W defined by $T_W(x, y) = \max(0, x + y - 1)$ for x, y in $[0, 1]$. On $\mathcal{L}^{\mathcal{I}}$, neither a BL-algebra nor an MV-algebra exists (as they are subclasses of MTL-algebras), yet in [21], it was proven that, for a t-norm T on [0,1]:

- T is divisible iff for each α in [0,1], $\mathcal{T}_{T,\alpha}$ is weakly divisible, that is, for x, y in L^I,
$$\mathcal{T}(x, \mathcal{I}_{\mathcal{T}}(x, y)) \bigsqcup \mathcal{T}(y, \mathcal{I}_{\mathcal{T}}(y, x)) = x \bigsqcap y$$
- T is involutive iff $\mathcal{T}_{T,0}$ is involutive, hence iff the pseudo t-representable t-norm corresponding to T is involutive; for $\alpha > 0$, $\mathcal{T}_{T,\alpha}$ is never involutive.
- $([0, 1], \min, \max, T, I_T, 0, 1)$ is an MV-algebra iff $(L^I, \bigsqcap, \bigsqcup, \mathcal{T}_{T,0}, \mathcal{I}_{\mathcal{T}_{T,0}}, [0, 0], [1, 1])$ is an involutive, weakly divisible residuated lattice.

BL-algebras and MV-algebras are quintessential in formal fuzzy logics as the algebraic counterparts to Basic Logic BL and Łukasiewicz logic Ł (see Section 1). The above results suggest that, in refining the conditions of triangle algebras (which play the same role for $\mathcal{L}^I$ as MTL-algebras do for [0,1], i.e. they characterize the residuated t-norms) to obtain more powerful structures, we should replace divisibility by weak divisibility.

Note also that the t-norm $\mathcal{T}_{T_W,0}$ on $\mathcal{L}^I$, which seems to satisfy the most useful properties (residuated, weakly divisible, involutive) is not t-representable. At this point, it remains an open question whether every weakly divisible, involutive triangle algebra on $\mathcal{L}^I$ is isomorphic to the triangle algebra induced by $\mathcal{T}_{T_W,0}$.

6 Conclusion and Future Work

In this paper, we established triangle algebras as the variety of interval-valued residuated lattices, in a similar way as MTL-algebras are the variety of prelinear residuated lattices. For our future work, we will use this crucial result to chart the landscape of fuzzy formal logics beyond prelinearity, and, more specifically, to develop a logic that formally characterizes tautologies (true formulas) in interval-valued residuated lattices. Concretely, a follow-up paper introducing "Triangle Logic" (TL) and proving its soundness and completeness w.r.t. triangle algebras is in preparation. Later on, new properties will be imposed on triangle algebras (and corresponding new axioms added to TL) to obtain more specific structures. The intention of these new properties is, amongst others, to characterize (a part of) the class of t-norms on $\mathcal{L}^I$ defined by Formula (3).
Another challenge for the future is to find out if TL (possibly enriched with more axioms) is standard complete, i.e. complete with respect to the corresponding triangle algebras on $\mathcal{L}^I$. In combination with a characterization of these triangle algebras, this would establish a logical calculus for interval truth values that is easy to handle and suitable for use in practical applications.
Furthermore also the links with other, comparable theories (see e.g. [4,10,20]) will be the subject of further research.

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