# Classification of Intuitionistic Fuzzy Implicators: an Algebraic Approach

**Chris Cornelis, Glad Deschrijver, Etienne E. Kerre**

Department of Mathematics and Computer Science, Ghent University
Fuzziness and Uncertainty Modelling Research Unit
Krijgslaan 281 (S9), B-9000 Gent, Belgium
E-mail: {`chris.cornelis,glad.deschrijver,etienne.kerre`}`@rug.ac.be`
WWW homepage: `http://fuzzy.rug.ac.be`

## Abstract

An attempt is made to build up a rigorous framework for classifying intuitionistic fuzzy (IF) implicators. Our approach will be algebraic: we treat IF implicators as lattice–valued mappings, and we translate desirable logical properties (in casu, a generalized version of the Smets–Magrez axioms) into algebraic equations to be satisfied.

## 1 Introduction and Preliminary Definitions

IF set theory has important opportunities for the representation of vagueness and uncertainty (see e.g. [1] [2] [4]). Inspired by fuzzy set theory, we believe that an algebraic treatment—translating desirable logical properties into algebraic laws (or axioms) to be satisfied—can shed a new light on the domain by providing a systematic yardstick for measuring the use of such or such an operator. In this paper, we attempt a classification of the concept of IF implication; we will necessarily also be concerned with the more basic operations of negation, $t$–norm and $t$–conorm, but they were not the main focus of this study.

The idea behind our approach is to treat logical connectives as algebraic mappings. To describe the domain and codomain structure the partially ordered set $(L^*, \leq_{L^*})$ was introduced in [2]:

**Definition 1.1 (Partially ordered set $(L^*, \leq_{L^*})$)**

$$L^* = \{(x_1, x_2) \in [0,1]^2 \mid x_1 + x_2 \leq 1\}$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \wedge x_2 \geq y_2$$

It is easily verified that $(L^*, \leq_{L^*})$ is a complete lattice. By $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$ we denote its units.

We proceed to give formal definitions as algebraic mappings in $L^*$ for each of the concepts of IF negation, $t$–norm, $t$–conorm and implication.[1]

**Definition 1.2 (IF Negator)** *An IF negator is any decreasing $L^* \to L^*$ mapping $\mathcal{N}$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}, \mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x, \forall x \in L^*$, then $\mathcal{N}$ is called an involutive IF negator.*

**Definition 1.3 (IF Triangular Norm)** *An IF $t$–norm is any monotonous, commutative, associative $(L^*)^2 \to L^*$ mapping $\mathcal{T}$ satisfying $\mathcal{T}(1_{L^*}, x) = x$, for all $x \in L^*$.*

**Definition 1.4 (IF Triangular Conorm)** *An IF $t$–conorm is any monotonous, commutative, associative $(L^*)^2 \to L^*$ mapping $\mathcal{S}$ satisfying $\mathcal{S}(0_{L^*}, x) = x$, for all $x \in L^*$.*

[1] For notational simplicity, we follow the convention to use calligraphic letters for IF connectives and standard capitals for fuzzy connectives.

IF $t$–norms and $t$–conorms can easily be generated using their fuzzy counterparts, as the following claim proves: [2]

**Theorem 1.1** *Given a fuzzy $t$–norm $T$ and $t$–conorm $S$ satisfying $T(\alpha,\beta) \leq 1 - S(1-\alpha, 1-\beta)$ for all $\alpha, \beta \in [0,1]$, the mappings $\mathcal{T}$ and $\mathcal{S}$ defined by, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $L^*$:*

$$\begin{aligned} \mathcal{T}(x,y) &= (T(x_1,y_1), S(x_2,y_2)), \\ \mathcal{S}(x,y) &= (S(x_1,y_1), T(x_2,y_2)), \end{aligned}$$

*are an IF $t$–norm and an IF $t$–conorm, respectively.*

Unfortunately, we do not have that for every IF $t$–norm (IF $t$–conorm) there exist a fuzzy $t$–norm and $t$–conorm such that the above equalities hold. We therefore need to introduce an additional definition:

**Definition 1.5 ($t$–representability)** *An IF t–norm $\mathcal{T}$ is called t–representable if there exist a fuzzy t–norm $T$ and fuzzy t–conorm $S$ such that $\mathcal{T}(x,y) = (T(x_1,y_1), S(x_2,y_2))$ for all $x = (x_1,x_2)$ and $y = (y_1,y_2)$ in $L^*$. An IF t–conorm $\mathcal{S}$ is called t–representable if there exist a fuzzy t–norm $T$ and fuzzy t–conorm $S$ such that $\mathcal{S}(x,y) = (S(x_1,y_1), T(x_2,y_2))$ for all $x = (x_1,x_2)$ and $y = (y_1,y_2)$ in $L^*$. In both cases, we say that $T$ and $S$ are representants of the IF t–(co)norm.*

For instance, it can be checked that the $(L^*)^2 \to L^*$ mapping $\mathcal{S}$ defined as, for $x = (x_1,x_2), y = (y_1,y_2) \in L^*$:

$$\mathcal{S}(x,y) = \begin{cases} x & \text{if } y = 0_{L^*} \\ y & \text{if } x = 0_{L^*} \\ (\max(1-x_2, 1-y_2), & \\ \min(x_2,y_2)) & \text{else} \end{cases}$$

is a non t–representable IF $t$–conorm.

**Definition 1.6 (IF Implicator)** *An IF implicator is any $(L^*)^2 \to L^*$–mapping $\mathcal{I}$ satisfying $\mathcal{I}(0_{L^*}, 0_{L^*}) = 1_{L^*}, \mathcal{I}(1_{L^*}, 0_{L^*}) = 0_{L^*}, \mathcal{I}(0_{L^*}, 1_{L^*}) = 1_{L^*}, \mathcal{I}(1_{L^*}, 1_{L^*}) = 1_{L^*}$. Moreover we require $\mathcal{I}$ to be decreasing in its first, and increasing in its second component.*

In the remainder of this section we introduce IF S– and R–implicators. It is easily verified that each of the mappings defined hereafter is an IF implicator in the sense of definition 1.6.

**Definition 1.7 (IF S–implicator)** *Let $\mathcal{S}$ be an IF t–conorm and $\mathcal{N}$ an IF negator. The IF S–implicator generated by $\mathcal{S}$ and $\mathcal{N}$ is the mapping $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ defined as, for $x, y \in L^*$ :*

$$\mathcal{I}_{\mathcal{S},\mathcal{N}}(x,y) = \mathcal{S}(\mathcal{N}(x), y)$$

*If $\mathcal{S}$ is t–representable, $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is called a t–representable IF S–implicator.*

**Definition 1.8 (IF R–implicator)** *Let $\mathcal{T}$ be an IF t–norm. The IF R–implicator generated by $\mathcal{T}$ is the mapping $\mathcal{I}_{\mathcal{T}}$ defined as, for $x, y \in L^*$ :*

$$\mathcal{I}_{\mathcal{T}}(x,y) = \sup\{\gamma \in L^* \mid \mathcal{T}(x,\gamma) \leq_{L^*} y\}$$

*If $\mathcal{T}$ is t–representable, $\mathcal{I}_{\mathcal{T}}$ is called a t–representable IF R–implicator.*

## 2 The axioms of Smets and Magrez

In [5], Smets and Magrez outlined an axiom scheme for fuzzy implicators. They took a number of tautologies from classical logic and translated them into an algebraic form. The scheme stands as a yardstick to test the suitability of fuzzy implicators. It is therefore instructive to generalize them to IF implicators:

**Definition 2.1 (Axioms of Smets and Magrez for an IF implicator $\mathcal{I}$)**

*(A.1)* $(\forall y \in L^*)(\mathcal{I}(.,y)$ *is decreasing in* $L^*)$
$(\forall x \in L^*)(\mathcal{I}(x,.)$ *is increasing in* $L^*)$
(monotonicity laws)

*(A.2)* $(\forall x \in L^*)(\mathcal{I}(1_{L^*}, x) = x)$
(neutrality principle)

*(A.3)* $(\forall (x,y) \in (L^*)^2)(\mathcal{I}(x,y) = \mathcal{I}(\mathcal{N}(y), \mathcal{N}(x)))$ (contrapositivity w.r.t. an IF negator $\mathcal{N}$)

*(A.4)* $(\forall (x,y,z) \in (L^*)^3)(\mathcal{I}(x, \mathcal{I}(y,z)) = \mathcal{I}(y, \mathcal{I}(x,z)))$
(interchangeability principle)

*(A.5)* $(\forall (x,y) \in (L^*)^2)(x \leq_{L^*} y \iff \mathcal{I}(x,y) = 1_{L^*})$ (confinement principle)

*(A.6)* $\mathcal{I}$ *is continuous* (continuity)

An IF implicator additionally satisfying the neutrality principle (A.2) is called an IF border implicator. A contrapositive IF border implicator additionally satisfying the interchangeability principle (A.4) is called an IF model implicator.

**Definition 2.2 (IF left continuity)** *Let $F$ be an arbitrary $L^* \to L^*$ function, then $F$ is called IF left-continuous, iff, for any $x_0 = (x_{0,1}, x_{0,2}) \in L^*$,*

$$(\forall \epsilon > 0)(\exists \delta_1 > 0)(\exists \delta_2 > 0)(\forall x \in L^*)$$
$$(x_{0,1} - \delta_1 < x_1 \leq x_{0,1} \wedge x_{0,2} \leq x_2 < x_{0,2} + \delta_2 \Rightarrow d(F(x), F(x_0)) < \epsilon)$$

*$F$ is called strictly IF left-continuous iff, for any $x_0 = (x_{0,1}, x_{0,2}) \in L^*$,*

$$(\forall \epsilon > 0)(\exists \delta_1 > 0)(\exists \delta_2 > 0)(\forall x \in L^*)$$
$$(x_{0,1} - \delta_1 < x_1 < x_{0,1} \wedge x_{0,2} < x_2 < x_{0,2} + \delta_2 \Rightarrow d(F(x), F(x_0)) < \epsilon)$$

*where $d$ represents the induced Euclidean metric on $L^*$.*

In the following two theorems, the results of our study of IF S–implicators and R–implicators w.r.t. the generalized Smets–Magrez axioms are summarized:

**Theorem 2.1 (Axioms of Smets and Magrez for IF S–implicators)**

1. *$\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is a border implicator for every choice of $\mathcal{S}$ and $\mathcal{N}$.*
2. *$\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is a model implicator for every choice of $\mathcal{S}$ and involutive $\mathcal{N}$.*
3. *If $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is t–representable, then it does not satisfy (A.5) for any choice of $\mathcal{S}$ and $\mathcal{N}$.*
4. *$\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is continuous for every choice of continuous $\mathcal{S}$ and $\mathcal{N}$.*

**Theorem 2.2 (Axioms of Smets and Magrez for IF R–implicators)**

1. *$\mathcal{I}_{\mathcal{T}}$ is a border implicator for every choice of $\mathcal{T}$.*
2. *If $\mathcal{I}_{\mathcal{T}}$ is t–representable, then it does not satisfy (A.3) for any choice of $\mathcal{T}$ and $\mathcal{N}$.*
3. *$\mathcal{I}_{\mathcal{T}}$ satisfies (A.5) if and only if there exists for every $x \in L^*$ a sequence $(\alpha_n)_{n \in \mathbb{N}^*}$ in $L^* \setminus A$, where $A = \{(\gamma_1, \gamma_2) \in L^* \mid \gamma_2 = 0\}$ such that*
$$\lim_{n \to +\infty} \alpha_n = 1_{L^*} \quad \text{and} \quad \lim_{n \to +\infty} \mathcal{T}(x, \alpha_n) = x \tag{1}$$
*In particular, if $\mathcal{T}$ is strictly IF left-continuous, then $\mathcal{I}_{\mathcal{T}}$ satisfies (A.5).*
4. *If $\mathcal{I}_{\mathcal{T}}$ is t–representable by a continuous t–norm $T$ and a continuous t–conorm $S$, then it does not necessarily satisfy (A.6).*

Theorem 2.1 tells us that every IF S–implicator is also an IF model implicator. It is less straightforward to see that the converse implication also holds, i.e. every IF model implicator is an IF S–implicator. To this effect the following lemmata are introduced.

**Lemma 2.1 (IF negator induced by an IF implicator)** *If $\mathcal{I}$ is an IF implicator, then the $L^* \to L^*$ mapping $\mathcal{N}_{\mathcal{I}}$ defined by, for $x \in L^*$,*

$$\mathcal{N}_{\mathcal{I}}(x) = \mathcal{I}(x, 0_{L^*})$$

*is an IF negator. We call $\mathcal{N}_{\mathcal{I}}$ the IF negator induced by $\mathcal{I}$.*

**Lemma 2.2** *Let $\mathcal{I}$ be an IF border implicator. If $\mathcal{I}$ is contrapositive w.r.t. $\mathcal{N}$, then $\mathcal{N} = \mathcal{N}_{\mathcal{I}}$, and in that case $\mathcal{N}_{\mathcal{I}}$ is involutive.*

**Lemma 2.3 (IF t–norm and t–conorm induced by an IF model implicator)** *If $\mathcal{I}$ is an IF model implicator, then the $L^{*2} \to L^*$ mappings $\mathcal{T}_\mathcal{I}$ and $\mathcal{S}_\mathcal{I}$ defined by, for $x, y \in L^*$,*

$$\begin{aligned} \mathcal{T}_\mathcal{I}(x,y) &= \mathcal{N}_\mathcal{I}(\mathcal{I}(x, \mathcal{N}_\mathcal{I}(y))) \\ \mathcal{S}_\mathcal{I}(x,y) &= \mathcal{I}(\mathcal{N}_\mathcal{I}(x), y) \end{aligned}$$

*are an IF t–norm and an IF t–conorm, respectively. We call $\mathcal{T}_\mathcal{I}$ the IF t–norm and $\mathcal{S}_\mathcal{I}$ the IF t–conorm induced by $\mathcal{I}$.*

**Definition 2.3 (IF De Morgan Quartet)** *An IF De Morgan quartet is any quartet $(\mathcal{T}, \mathcal{S}, \mathcal{N}, \mathcal{I})$ consisting of an IF t–norm $\mathcal{T}$, an IF t–conorm $\mathcal{S}$, an IF negator $\mathcal{N}$ and an IF implicator $\mathcal{I}$ such that, for all $x, y \in L^*$:*

$$\mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y))) = \mathcal{S}(x,y)$$

$$\mathcal{N}(\mathcal{S}(\mathcal{N}(x), \mathcal{N}(y))) = \mathcal{T}(x,y)$$

$$\mathcal{I}(x,y) = \mathcal{S}(\mathcal{N}(x), y)$$

**Theorem 2.3** *For an IF model implicator $\mathcal{I}$, $(\mathcal{T}_\mathcal{I}, \mathcal{S}_\mathcal{I}, \mathcal{N}_\mathcal{I}, \mathcal{I})$ is an IF De Morgan quartet.*

**Corollary 2.1** *An IF model implicator is an IF $\mathcal{S}$–implicator.*

**Corollary 2.2** *No t–representable IF implicator satisfies all Smets–Magrez axioms simultaneously.*

It is compelling to note that unless there exists a continuous IF S–implicator that is not $t$–representable and satisfies (A.5), no IF implicator satisfies all Smets–Magrez axioms at once. To this aim, one might try to prove that any IF S–implicator satisfying (A.5) cannot be continuous.

# 3 Conclusion

Basing ourselves on ideas from fuzzy set theory, we provided a yardstick method to test the suitability of intuitionistic fuzzy implicators, for use in e.g. the Intuitionistic Compositional Rule of Inference. [2] [4]

As a byproduct, we introduced a novel order–theoretic definition of IF t–norms and IF t–conorms, as opposed to the classical way of defining connectives in IFS theory, that is: by taking as a point of departure definitions of connectives for fuzzy sets and extending them. A future paper [3] will deal with this matter in more detail.

# Acknowledgements

Chris Cornelis would like to thank the Fund for Scientific Research Flanders (FWO) for funding the research elaborated on in this paper.

# References

[1] K. T. ATANASSOV, **Intuitionistic fuzzy sets**, Physica–Verlag, Heidelberg – New York, (1999)

[2] C. CORNELIS, G. DESCHRIJVER, **The Compositional Rule of Inference in an Intuitionistic Fuzzy Logic Framework**, Proceedings of ESSLLI 2001 Student Session, (Kristina Striegnitz, ed.), Kluwer Academic Publishers, (2001), 83–94

[3] G. DESCHRIJVER, C. CORNELIS, E. E. KERRE, **Intuitionistic Fuzzy Connectives Revisited**, In preparation, (2001)

[4] C. CORNELIS, E. E. KERRE, **Reliability of Information: Motivation for an Intuitionistic Theory of Possibility**, Submitted to Fuzzy Sets and Systems, (2001)

[5] P. SMETS, P. MAGREZ, **Implication in fuzzy logic**, Internat. J. Approximate Reasoning, **1**, (1987), 327–347