A characterization of interval-valued residuated lattices

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Abstract

As is well-known, residuated lattices (RLs) on the unit interval correspond to left-continuous t-norms. Thus far, a similar characterization has not been found for RLs on the set of intervals of [0,1], or more generally, of a bounded lattice $\mathcal{L}$. In this paper, we show that the open problem can be solved if it is restricted, making only a few simple and intuitive assumptions, to the class of interval-valued residuated lattices (IVRLs).

More specifically, we derive a full characterization of product and implication in IVRLs in terms of their counterparts on the base RL. To this aim, we use triangle algebras, a recently introduced variety of RLs that serves as an equational representation of IVRLs.

Key words:
interval-valued fuzzy set theory, residuated lattices, triangle algebras
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1 Introduction

In the original, and still most popular, approach to fuzzy set theory introduced by Zadeh [26], membership values are drawn from the unit interval, equipped with the usual ordering, and intersection and union are modeled by minimum

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and maximum, respectively. On the other hand, many useful generalizations of the evaluation structure as well as of the connectives have been proposed. Goguen [14] replaced the structure of the totally ordered unit interval by an arbitrary bounded lattice $\mathcal{L}$ to allow for incomparabilities among elements, and triangular norms and conorms [21] are quite common nowadays as generalized representations of logical conjunction and disjunction, respectively. Remark that their definition, originally introduced on $([0,1], \min, \max)$, can be naturally extended to a bounded lattice $\mathcal{L}$, by virtue of the ordering of $\mathcal{L}$.

Triangular norms (t-norms for short) are often classified based on the properties they satisfy (see e.g. [19]). From a logical point of view, a particularly useful property for a t-norm is the residuation principle, i.e., the existence of an implication $\mathcal{I}_{\mathcal{T}}$ (called residual implication) satisfying $\mathcal{T}(x, y) \leq z$ iff $x \leq \mathcal{I}_{\mathcal{T}}(y, z)$, for all possible x, y and z in $\mathcal{L}$; the corresponding logical structure is a residuated lattice (RL). On the unit interval, the t-norms that satisfy this property are exactly those that are left-continuous. In general, however, no such easy characterizations exist.

Here, we focus on triangularizations, i.e., lattices on the set of intervals of a bounded lattice $\mathcal{L}$ (the base lattice). Such triangularizations are often used to model not only gradedness, but also a degree of imprecision or uncertainty. The best-known instance is probably $\mathcal{L}^I$, which emerges by taking $\mathcal{L} = ([0,1], \min, \max)$ as the base lattice. The study of t-norms on $\mathcal{L}^I$ satisfying the residuation principle has received ample attention (see e.g. [5,7,8]), but their characterization currently still remains an important unsolved problem.

In its most general form, the open problem is complicated by the fact that certain "badly behaved" t-norms may still satisfy the residuation principle (see, e.g., [7] and [18]). For such a t-norm $\mathcal{T}$ on $\mathcal{L}^I$, there exist x, y in $[0,1]$ such that either $\mathcal{T}([x,x],[y,y]) = [z_1, z_2]$, or $\mathcal{I}_{\mathcal{T}}([x,x],[y,y]) = [z_1, z_2]$ with $z_1 < z_2$. In other words, "exact" inputs (i.e., degenerated intervals) are not necessarily mapped to an exact output. Many authors (e.g. Alcalde et al. in [1], Bustince in [2], Esteva et al. in [11], Gehrke et al. in [13] and Höhle in [16]) therefore do not consider these "badly behaved" t-norms.

If we restrict such operations from consideration, the corresponding logical structures are interval-valued residuated lattices (IVRLs) [24]. In [25], IVRLs were shown to be isomorphic to triangle algebras, a variety of RLs equipped with approximation operators, and with a third angular point u (uncertainty), different from 0 and 1. As such, triangle algebras provide an equational representation (i.e, strictly in terms of identities) of IVRLs, which makes them a very productive aid in establishing formal results about IVRLs. In [25], for instance, they were used to construct Triangle Logic (TL), a formal fuzzy logic that was shown to be sound and complete with respect to the class of IVRLs.

In this paper, we use the isomorphism with triangle algebras to derive the general form of product and implication in IVRLs. Apart from solving the above-mentioned open problem, this result also allows to obtain characterizations for other important properties in IVRLs, including involutivity and divisibility.

The remainder of this article is structured as follows: Section 2 proceeds with a recapitulation of all required definitions and properties. The main result is then presented in Section 3, which also contains the technical exposition in terms of triangle algebras that leads up to its proof, as well as some of its corollaries. We then investigate the particular situation on the "standard" IVRLs, i.e., those built on top of $\mathcal{L}^I$, in Section 4. We end with a conclusion and some ideas for future work.

2 Preliminaries

Definition 1 *A residuated lattice*[2] *[10] is a structure* $\mathcal{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ *in which* $\sqcap, \sqcup, *$ *and* $\Rightarrow$ *are binary operators on the set* L *and*

- $(L, \sqcap, \sqcup)$ *is a bounded lattice with* 0 *as smallest and* 1 *as greatest element,*
- $*$ *is commutative and associative, with* 1 *as neutral element, and*
- $x * y \leq z$ *iff* $x \leq y \Rightarrow z$ *for all* x, y *and* z *in* L *(residuation principle).*

The binary operations $*$ and $\Rightarrow$ are called product and implication, respectively. We will use the notations $\neg x$ for $x \Rightarrow 0$ (negation), $x \Leftrightarrow y$ for $(x \Rightarrow y) \sqcap (y \Rightarrow x)$ and x^n for $\underbrace{x * x * \ldots * x}_{n \text{ times}}$.

Proposition 2 *In a residuated lattice* $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$*,* $*$ *is increasing in both arguments and* $\Rightarrow$ *is decreasing in the first and increasing in the second argument. Furthermore the following inequalities and identities hold, for every* x*,* y *and* z *in* L*:*

[2] In literature (e.g. in [16]), the name residuated lattice is sometimes used for structures more general than what we call residuated lattices. In the most general terminology, our structures would be called bounded integral commutative residuated lattices.

$$x * y \leq x \sqcap y \quad (1)$$
$$y \leq x \Rightarrow y \quad (2)$$
$$x \leq y \Rightarrow (x * y) \quad (3)$$
$$x * (x \Rightarrow y) \leq x \sqcap y \quad (4)$$
$$x \sqcup y \leq (x \Rightarrow y) \Rightarrow y \quad (5)$$
$$x \Rightarrow (y \sqcap z) = (x \Rightarrow y) \sqcap (x \Rightarrow z) \quad (6)$$
$$(x \sqcup y) \Rightarrow z = (x \Rightarrow z) \sqcap (y \Rightarrow z) \quad (7)$$
$$(x * y) \Rightarrow z = x \Rightarrow (y \Rightarrow z) \quad (8)$$

The proofs can be found in, e.g., [22].

Definition 3 *An MTL-algebra [12] is a prelinear residuated lattice, i.e., a residuated lattice in which* $(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$ *for all* x *and* y *in* L.
A BL-algebra [15] is a divisible MTL-algebra, i.e., an MTL-algebra in which $x \sqcap y = x * (x \Rightarrow y)$ *for all* x *and* y *in* L. *The weaker property* $x \sqcap y = (x * (x \Rightarrow y)) \sqcup (y * (y \Rightarrow x))$ *is called weak divisibility [23,25].*
An MV-algebra [3,4] is a BL-algebra in which the negation is an involution, i.e., $(x \Rightarrow 0) \Rightarrow 0 = x$ *for all* x *in* L.
A Heyting-algebra, or pseudo-Boolean algebra [20], is a residuated lattice in which $x * x = x$ *for all* x *in* L, *or, equivalently, in which* $x * y = x \sqcap y$ *for all* x *and* y *in* L.
A Boolean algebra [17] is an MV-algebra that is also a Heyting-algebra.

In residuated lattices, divisibility is equivalent to the following property: if $x \leq y$, then there exists a z such that $x = y * z$ (see, e.g., [16]). Using this equivalence, it is easy to see that a Heyting algebra is always divisible. Being divisible residuated lattices, Heyting algebras are always distributive [16].

Definition 4 *A triangular norm (t-norm, for short) [21] on a bounded lattice* $(L, \sqcap, \sqcup, 0, 1)$ *is a binary, increasing, commutative and associative operator* $T : L^2 \to L$ *that satisfies* $T(x, 1) = 1$, *for all* x *in* L.
If for every pair (x, y) *in* L^2, $\sup\{z \in L \mid T(x, z) \leq y\}$ *exists, then the map* I_T *defined by* $I_T(x, y) = \sup\{z \in L : T(x, z) \leq y\}$ *is called the residual implicator of* T.

In a residuated lattice, the operator $*$ is always a t-norm, and $\Rightarrow$ is its residual implicator. Moreover, for a residuated lattice on the unit interval, $*$ is always left-continuous (see e.g. [12]).

Definition 5 Given a lattice $\mathcal{L} = (L, \sqcap, \sqcup)$ (called the base lattice), its triangularization $\mathbb{T}(\mathcal{L})$ is the structure $\mathbb{T}(\mathcal{L}) = (Int(\mathcal{L}), \bigsqcap, \bigsqcup)$ defined by

- $Int(\mathcal{L}) = \{[x_1, x_2] \mid (x_1, x_2) \in L^2 \text{ and } x_1 \leq x_2\}$

- $[x_1, x_2] \bigsqcap [y_1, y_2] = [x_1 \sqcap y_1, x_2 \sqcap y_2]$
- $[x_1, x_2] \bigsqcup [y_1, y_2] = [x_1 \sqcup y_1, x_2 \sqcup y_2]$

The set $D_{\mathcal{L}} = \{[x, x] \mid x \in L\}$ is called the diagonal of $\mathbb{T}(\mathcal{L})$. The first and the second projection pr_1 and pr_2 are the mappings from $Int(\mathcal{L})$ to L, defined by $pr_1([x_1, x_2]) = x_1$ and $pr_2([x_1, x_2]) = x_2$, for all $[x_1, x_2]$ in $T(\mathcal{L})$.

The triangularization of $([0, 1], \min, \max)$ is denoted as $\mathcal{L}^I = (L^I, \sqcap, \sqcup)$ (thus, $L^I = Int([0, 1], \min, \max)$ and $\mathcal{L}^I = \mathbb{T}([0, 1], \min, \max))$ and is shown in Figure 1.

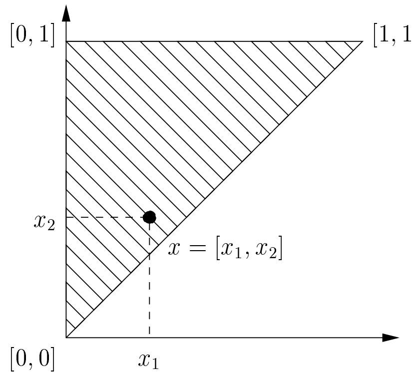


Fig. 1. The lattice $\mathcal{L}^I$

Definition 6 An interval-valued residuated lattice (IVRL) [25] is a residuated lattice $(Int(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0, 0], [1, 1])$ on the triangularization $\mathbb{T}(\mathcal{L})$ of a bounded lattice $\mathcal{L} = (L, \sqcap, \sqcup)$, in which the diagonal $D_{\mathcal{L}}$ is closed under $\odot$ and $\Rightarrow_\odot$, i.e., $[x_1, x_1] \odot [y_1, y_1] \in D_{\mathcal{L}}$ and $[x_1, x_1] \Rightarrow_\odot [y_1, y_1] \in D_{\mathcal{L}}$ for all x_1, y_1 in L.

Henceforth, an IVRL on $\mathcal{L}^I$ will be called a *standard IVRL*.

Example 7 *If T is a left-continuous t-norm on $([0, 1], \min, \max)$, $\alpha \in [0, 1]$ and the mapping $\mathcal{T}_{T,\alpha}$ [7] is defined, for $x = [x_1, x_2]$ and $y = [y_1, y_2]$ in L^I, by the formula*

$$\mathcal{T}_{T,\alpha}(x, y) = [T(x_1, y_1), \max(T(\alpha, T(x_2, y_2)), T(x_1, y_2), T(x_2, y_1))], \tag{9}$$

then $(L^I, \sqcap, \sqcup, \mathcal{T}_{T,\alpha}, \mathcal{I}_{\mathcal{T}_{T,\alpha}}, [0, 0], [1, 1])$ is a standard IVRL, in which the residual implicator $\mathcal{I}_{\mathcal{T}_{T,\alpha}}$ of $\mathcal{T}_{T,\alpha}$ is given by:

$$\mathcal{I}_{\mathcal{T}_{T,\alpha}}(x, y) = [\min(I_T(x_1, y_1), I_T(x_2, y_2)), \min(I_T(T(x_2, \alpha), y_2), I_T(x_1, y_2))].$$

This construction can be easily generalized for an arbitrary residuated lattice

$(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ *and its triangularization, similarly as in [25]. If* $\alpha = 1$, $\mathcal{T}_{T,\alpha}$ *is called t-representable, and if* $\alpha = 0$, *then* $\mathcal{T}_{T,\alpha}$ *is called pseudo t-representable [8]. Remark also that* $\alpha = pr_2(\mathcal{T}_{T,\alpha}([0,1],[0,1]))$.

In [24], we introduced the notion of a triangle algebra, a structure that serves as an equational representation for an interval-valued residuated lattice.

Definition 8 A triangle algebra is a structure $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, in which $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice, ν and μ are unary operators and u a constant, such that the following conditions are satisfied:

$T.1$ $\nu x \leq x$,

$T.1'$ $x \leq \mu x$,

$T.2$ $\nu x \leq \nu\nu x$,

$T.2'$ $\mu\mu x \leq \mu x$,

$T.3$ $\nu(x \sqcap y) = \nu x \sqcap \nu y$,

$T.3'$ $\mu(x \sqcap y) = \mu x \sqcap \mu y$,

$T.4$ $\nu(x \sqcup y) = \nu x \sqcup \nu y$,

$T.4'$ $\mu(x \sqcup y) = \mu x \sqcup \mu y$,

$T.5$ $\nu u = 0$,

$T.5'$ $\mu u = 1$,

$T.6$ $\nu\mu x = \mu x$,

$T.6'$ $\mu\nu x = \nu x$,

$T.7$ $\nu(x \Rightarrow y) \leq \nu x \Rightarrow \nu y$,

$T.8$ $(\nu x \Leftrightarrow \nu y) * (\mu x \Leftrightarrow \mu y) \leq (x \Leftrightarrow y)$,

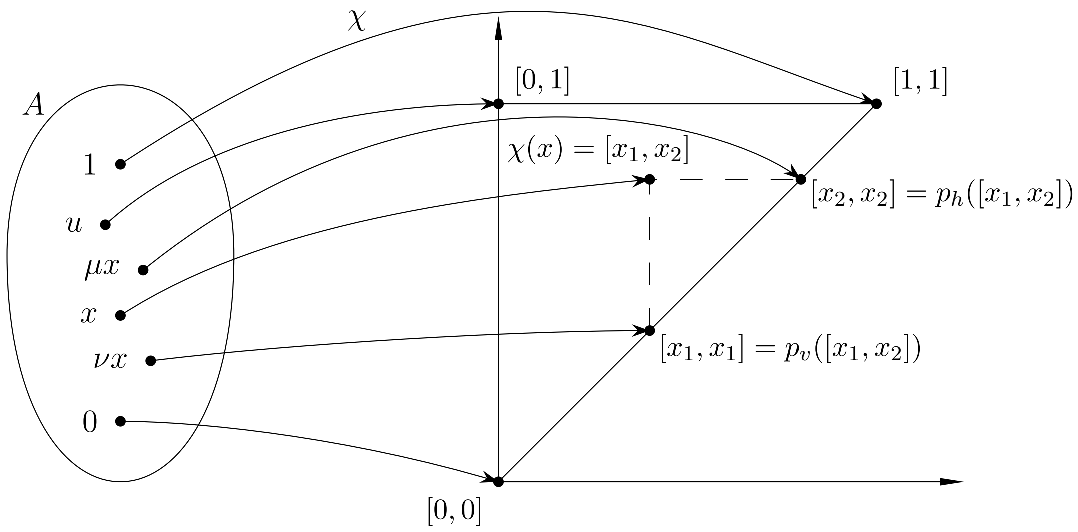
$T.9$ $\nu x \Rightarrow \nu y \leq \nu(\nu x \Rightarrow \nu y)$.

Remark that this definition is not exactly the same as the one we gave in [**?**]. In that article we added the conditions $\nu 1 = 1$ and $\mu 0 = 0$. However, they can be left out because they can be proven from the other conditions. Indeed, using T.5' and T.6, we find $\nu 1 = \nu\mu u = \mu u = 1$; and, using T.5 and T.6', $\mu 0 = \mu\nu u = \nu u = 0$.

In a triangle algebra $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, the unary operators ν and μ are increasing. In the remainder of the paper, we will use the abbreviation (M) whenever we refer to the monotonicity of $*$ and $\Rightarrow$ (see Proposition 2), and of ν and μ.
Another property valid in triangle algebras is $x \leq y$ iff $x \Rightarrow y = 1$ iff $\nu x \leq \nu y$ and $\mu x \leq \mu y$ (characterization of inequality). This will be abbreviated by (I).

Definition 9 *Let* $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ *be a triangle algebra. An element* x *in* A *is called exact if* $\nu x = x$. *The set of exact elements of* $\mathcal{A}$ *is denoted by* $E(\mathcal{A})$.

In [25], it was proven that $E(\mathcal{A}) = \nu(A) = \mu(A) = \nu(E(\mathcal{A})) = \mu(E(\mathcal{A}))$ and that $(E(\mathcal{A}), \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is an algebraic subreduct of $\mathcal{A}$. So $E(\mathcal{A})$ is closed



Triangle algebra
$(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$

Isomorphic triangle algebra
$(A', \sqcap', \sqcup', *', \Rightarrow', p_v, p_h, [0,0], [0,1], [1,1])$
in which $(A', \sqcap', \sqcup', *', \Rightarrow', [0,0], [1,1])$
is an IVRL

Fig. 2. The isomorphism χ from a triangle algebra to an IVRL.

under $*$ and $\Rightarrow$, i.e., $\nu(x * y) = x * y$ and $\nu(x \Rightarrow y) = x \Rightarrow y$ if $\nu x = x$ and $\nu y = y$. In the remainder of this paper, we will use (C) to refer to this property.

We also established in [25] the one-to-one correspondence between IVRLs and triangle algebras. The unary operators ν and μ correspond with the mappings p_v and p_h that map $[x_1, x_2]$ to $[x_1, x_1]$ and $[x_2, x_2]$ respectively. The constant u corresponds to $[0, 1]$, and the set of exact elements $E(\mathcal{A})$ to the diagonal $D_{\mathcal{L}}$ (see Figure 2).

Every property mentioned in Definition 3 (prelinearity, divisibility, ...) can be weakened, by imposing it on $E(\mathcal{A})$ (instead of A) only. We will denote this with the prefix 'pseudo'. For example, a triangle algebra is said to be pseudo-divisible if $\nu x \sqcap \nu y = \nu x * (\nu x \Rightarrow \nu y)$ for all x and y in A. Another example: a triangle algebra is said to be pseudo-linear if its set of exact elements is linearly ordered (by the original ordering, restricted to $E(\mathcal{A})$). There are infinitely many pseudo-linear triangle algebras (e.g. each of those defined by the t-norms on $\mathcal{L}^I$ from Example 7), but only those with less than four elements are linear too. This is due to the fact that, if a triangle algebra has at least four elements, then there is at least one exact element different from 0 and 1. Indeed, if 0 and 1 were the only exact elements, then the only possible couples $(\nu x, \mu x)$ would be $(0, 0)$, $(0, 1)$ and $(1, 1)$. Condition T.8 implies that the triangle algebra has three elements in that case (0, u and 1), a contradiction. Using (I), it is easy to see that an exact element different from 0 and 1 is incomparable with u.

We will use the same terminology for IVRLs as for triangle algebras. For example: an IVRL is pseudo-divisible if the structure, restricted to the diagonal, is a divisible residuated lattice. Another example: $\mathcal{L}^I$ is pseudo-linear, as its diagonal is linear, but $\mathcal{L}^I$ is clearly not linear itself (consider, e.g., $[0,1]$ and $[0.5,0.5]$).

To conclude this section we recall some basic properties of triangle algebras [25].

Proposition 10 *In a triangle algebra $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, the following identities and inequality hold, for every x, y and z in A:*

$$x \sqcap u = \mu x \sqcap u \tag{10}$$
$$x \sqcup u = \nu x \sqcup u \tag{11}$$
$$x = (\mu x \sqcap u) \sqcup \nu x = (\nu x \sqcup u) \sqcap \mu x \tag{12}$$
$$\nu(x * y) = \nu x * \nu y \tag{13}$$
$$\mu(x * y) \leq \mu x * \mu y \tag{14}$$

PROOF. We prove (10), (11) and (12) (the other two proofs can be found in [25]):

- On the one hand, we have $\nu(x \sqcap u) = \nu x \sqcap \nu u = 0 = \nu\mu x \sqcap \nu u = \nu(\mu x \sqcap u)$. On the other hand, we have $\mu(x \sqcap u) = \mu x \sqcap \mu u = \mu\mu x \sqcap \mu u = \mu(\mu x \sqcap u)$. So, by T.8, $x \sqcap u = \mu x \sqcap u$.
- The proof of $x \sqcup u = \nu x \sqcup u$ is analogous to that of $x \sqcap u = \mu x \sqcap u$.
- On the one hand, we can use T.4, T.3, T.1 and T.2 to find $\nu((x \sqcap u) \sqcup \nu x) = (\nu x \sqcap \nu u) \sqcup \nu x = \nu x$. On the other hand, by T.4', T.3', T.6', T.5', T.1 and T.1', we find $\mu((x \sqcap u) \sqcup \nu x) = (\mu x \sqcap \mu u) \sqcup \nu x = \mu x$. So by T.8, $(x \sqcap u) \sqcup \nu x = x$. Together with (10), this proves the first identity. The other identity is proven in a similar way.

□

3 Product and Implication

In this section, we will prove, by a series of steps leading up to Theorem 14, that the operations $*$ and $\Rightarrow$ in a triangle algebra $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ are determined by their action on $E(\mathcal{A})$ and by the value of $u * u$. To this aim, we will derive explicit representations for $\nu(x * y)$, $\mu(x * y)$, $\nu(x \Rightarrow y)$, and

$\mu(x \Rightarrow y)$. To obtain these, first we prove a number of properties regarding the interaction of the operations in triangle algebras.

Proposition 11 *In a triangle algebra $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, the following inequality and identity hold, for every x and y in A:*

$$\mu(x \Rightarrow y) \leq \nu x \Rightarrow \mu y \tag{15}$$
$$\mu(x * \nu y) = \mu x * \nu y \tag{16}$$

PROOF.

(15) Let x and y be in A. Because νx and μy are in $E(\mathcal{A})$, also $\nu x \Rightarrow \mu y \in E(\mathcal{A})$: $\nu x \Rightarrow \mu y = \mu(\nu x \Rightarrow \mu y)$ (C). Using $\nu x \leq x$ (T.1), $y \leq \mu y$ (T.1'), we find $x \Rightarrow y \leq \nu x \Rightarrow \mu y$ (M). Thus $\mu(x \Rightarrow y) \leq \mu(\nu x \Rightarrow \mu y)$ (M).

(16) By (14) and T.6', we already know $\mu(x * \nu y) \leq \mu x * \mu\nu y = \mu x * \nu y$. Furthermore, by (3), (M), (15) and T.2, $\mu x \leq \mu(\nu y \Rightarrow (x * \nu y)) \leq \nu\nu y \Rightarrow \mu(x * \nu y) = \nu y \Rightarrow \mu(x * \nu y)$. Therefore also $\mu x * \nu y \leq \mu(x * \nu y)$ by the residuation principle.

□

Remark that (16) means in fact that $\mu(x * z) = \mu x * z$ if $z \in E(\mathcal{A})$ (because in this case $z = \nu z$).

The following very useful lemma will be needed in the derivation of further identities in triangle algebras.

Lemma 12 *In a triangle algebra $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, for all y in A and all z in $E(\mathcal{A})$, it holds that $u * z \leq y$ iff $z \leq \mu y$.*

PROOF. Remark that $\mu(u * z) = \mu u * z = z$ because of (16) and T.5'. Suppose $z \leq \mu y$. This means $\mu(u * z) \leq \mu y$. As $\nu(u * z) = 0 \leq \nu y$ by (M) and T.5, we conclude that $u * z \leq y$ (using (I)). Conversely, if $u * z \leq y$, then $z = \mu(u * z) \leq \mu y$ by (M). □

Proposition 13 *In a triangle algebra $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, the following identities hold, for every x and y in A:*

$$\nu(\nu x \Rightarrow y) = \nu x \Rightarrow \nu y \tag{17}$$
$$\mu(\nu x \Rightarrow y) = \nu x \Rightarrow \mu y \tag{18}$$
$$(x \sqcap u) \Rightarrow (y \sqcap u) = (x \sqcap u) \Rightarrow y = (x \sqcap u) \Rightarrow \mu y \tag{19}$$
$$\nu((x \sqcap u) \Rightarrow y) = \mu x \Rightarrow \mu y \tag{20}$$
$$\mu((x \sqcap u) \Rightarrow y) = \mu x \Rightarrow \mu(u \Rightarrow y) \tag{21}$$
$$\mu(u \Rightarrow y) = \mu(u * u) \Rightarrow \mu y \tag{22}$$

PROOF.

(17) On the one hand, $\nu x \Rightarrow \nu y \leq \nu x \Rightarrow y$, by T.1 and (M), so $\nu x \Rightarrow \nu y = \nu(\nu x \Rightarrow \nu y) \leq \nu(\nu x \Rightarrow y)$ by (C) and (M). On the other hand, by T.1, $\nu(\nu x \Rightarrow y) \leq \nu x \Rightarrow y$. By the residuation principle, $\nu x * \nu(\nu x \Rightarrow y) \leq y$. Moreover, $\nu x * \nu(\nu x \Rightarrow y) = \nu(\nu x * \nu(\nu x \Rightarrow y)) \leq \nu y$ by (C) and (M). Applying the residuation principle again, we obtain $\nu(\nu x \Rightarrow y) \leq \nu x \Rightarrow \nu y$.

(18) We will prove this identity by showing that $z \leq \nu x \Rightarrow \mu y$ iff $z \leq \mu(\nu x \Rightarrow y)$ for any $z \in E(\mathcal{A})$. This implies $\nu x \Rightarrow \mu y = \mu(\nu x \Rightarrow y)$, because both elements in this identity are in $E(\mathcal{A})$. We have $z \leq \nu x \Rightarrow \mu y$ iff $z * \nu x \leq \mu y$ iff (by Lemma 12) $u * z * \nu x \leq y$ iff $u * z \leq \nu x \Rightarrow y$ iff (by Lemma 12) $z \leq \mu(\nu x \Rightarrow y)$.

(19) Because $y \leq \mu y$ and $\Rightarrow$ is increasing in the second argument, $(x \sqcap u) \Rightarrow y \leq (x \sqcap u) \Rightarrow \mu y$. Because $\nu((x \sqcap u) * ((x \sqcap u) \Rightarrow \mu y)) \leq \nu(x \sqcap u) \leq \nu u = 0 \leq \nu y$ and $\mu((x \sqcap u) * ((x \sqcap u) \Rightarrow \mu y)) \leq \mu\mu y = \mu y$ by (4) and (M), $(x \sqcap u) * ((x \sqcap u) \Rightarrow \mu y) \leq y$, by (I) and T.8. Thus $(x \sqcap u) \Rightarrow \mu y \leq (x \sqcap u) \Rightarrow y$. So $(x \sqcap u) \Rightarrow \mu y = (x \sqcap u) \Rightarrow y$.
Now we apply this with $y \sqcap u$ instead of y, using $\mu y = \mu(y \sqcap u)$. We find $(x \sqcap u) \Rightarrow \mu y = (x \sqcap u) \Rightarrow \mu(y \sqcap u) = (x \sqcap u) \Rightarrow (y \sqcap u)$.

(20) Because $(x \sqcap u) \Rightarrow \mu y = (x \sqcap u) \Rightarrow y$ by (19), we know that $\nu((x \sqcap u) \Rightarrow \mu y) = \nu((x \sqcap u) \Rightarrow y)$. We now prove that $\nu((x \sqcap u) \Rightarrow \mu y) = \mu x \Rightarrow \mu y$. On the one hand, $\mu x \Rightarrow \mu y \leq (x \sqcap u) \Rightarrow \mu y$ because $x \sqcap u \leq x \leq \mu x$ and (M). Thus, by (C) and (M), $\mu x \Rightarrow \mu y = \nu(\mu x \Rightarrow \mu y) \leq \nu((x \sqcap u) \Rightarrow \mu y)$. On the other hand, we have $(x \sqcap u) * \nu((x \sqcap u) \Rightarrow \mu y) \leq (x \sqcap u) * ((x \sqcap u) \Rightarrow \mu y) \leq \mu y$ because of T.1, (M) and (4). Therefore, using T.3', T.5', (16), (M) and T.2', $\mu x * \nu((x \sqcap u) \Rightarrow \mu y) = \mu(x \sqcap u) * \nu((x \sqcap u) \Rightarrow \mu y) = \mu((x \sqcap u) * \nu((x \sqcap u) \Rightarrow \mu y)) \leq \mu\mu y = \mu y$, which implies $\nu((x \sqcap u) \Rightarrow \mu y) \leq \mu x \Rightarrow \mu y$.

(21) We will prove this identity by showing that $z \leq \mu((x \sqcap u) \Rightarrow y)$ iff $z \leq \mu x \Rightarrow \mu(u \Rightarrow y)$ for any $z \in E(\mathcal{A})$. This implies $\mu((x \sqcap u) \Rightarrow y) = \mu x \Rightarrow \mu(u \Rightarrow y)$, because both elements in this identity are in $E(\mathcal{A})$.
Using Lemma 12, the residuation principle, (I), $\nu(z * (x \sqcap u)) = 0$, (16), T.3' and T.5', we find $z \leq \mu((x \sqcap u) \Rightarrow y)$ iff $u * z \leq (x \sqcap u) \Rightarrow y$ iff $z * (x \sqcap u) \leq u \Rightarrow y$ iff $\mu(z * (x \sqcap u)) \leq \mu(u \Rightarrow y)$ iff $z * \mu(x \sqcap u) \leq \mu(u \Rightarrow y)$ iff $z * \mu x \leq \mu(u \Rightarrow y)$ iff $z \leq \mu x \Rightarrow \mu(u \Rightarrow y)$.

(22) We will prove this identity by showing that $z \leq \mu(u * u) \Rightarrow \mu y$ iff $z \leq \mu(u \Rightarrow y)$ for any $z \in E(\mathcal{A})$. This implies $\mu(u * u) \Rightarrow \mu y = \mu(u \Rightarrow y)$, because both sides in this identity are in $E(\mathcal{A})$. Applying the residuation principle, (16), (I), $\nu(z * u * u) = 0$ and Lemma 12, we find the following equivalences: $z \leq \mu(u * u) \Rightarrow \mu y$ iff $z * \mu(u * u) \leq \mu y$ iff $\mu(z * u * u) \leq \mu y$ iff $z * u * u \leq y$ iff $z * u \leq u \Rightarrow y$ iff $z \leq \mu(u \Rightarrow y)$.

$\square$

Theorem 14 *In a triangle algebra $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, the implication $\Rightarrow$ and the product $*$ are completely determined by their action on $E(\mathcal{A})$ and the value of $u * u$. More specifically:*

- $\nu(x \Rightarrow y) = (\nu x \Rightarrow \nu y) \sqcap (\mu x \Rightarrow \mu y)$
- $\mu(x \Rightarrow y) = (\mu x \Rightarrow (\mu(u * u) \Rightarrow \mu y)) \sqcap (\nu x \Rightarrow \mu y)$
- $\nu(x * y) = \nu x * \nu y$
- $\mu(x * y) = (\nu x * \mu y) \sqcup (\mu x * \nu y) \sqcup (\mu x * \mu y * \mu(u * u))$

and therefore

$$\begin{aligned} &x \Rightarrow y \\ &= \left(\left(\left(\mu x \Rightarrow (\mu(u * u) \Rightarrow \mu y)\right) \sqcap (\nu x \Rightarrow \mu y)\right) \sqcap u\right) \sqcup \left((\mu x \Rightarrow \mu y) \sqcap (\nu x \Rightarrow \nu y)\right). \\ &= \left(((\mu x \Rightarrow \mu y) \sqcap (\nu x \Rightarrow \nu y)) \sqcup u\right) \sqcap \left(\left(\mu x \Rightarrow (\mu(u * u) \Rightarrow \mu y)\right) \sqcap (\nu x \Rightarrow \mu y)\right) \end{aligned}$$

and

$$\begin{aligned} x * y &= \left(\left((\nu x * \mu y) \sqcup (\mu x * \nu y) \sqcup (\mu x * \mu y * \mu(u * u))\right) \sqcap u\right) \sqcup (\nu x * \nu y). \\ &= \left((\nu x * \nu y) \sqcup u\right) \sqcap \left((\nu x * \mu y) \sqcup (\mu x * \nu y) \sqcup (\mu x * \mu y * \mu(u * u))\right) \end{aligned}$$

PROOF.

- By means of (12), (10) and (7), we find

$$x \Rightarrow y = ((x \sqcap u) \sqcup \nu x) \Rightarrow y = ((x \sqcap u) \Rightarrow y) \sqcap (\nu x \Rightarrow y).$$

Therefore, by T.3, (20) and (17),

$$\begin{aligned} \nu(x \Rightarrow y) &= \nu(((x \sqcap u) \Rightarrow y) \sqcap (\nu x \Rightarrow y)) \\ &= \nu((x \sqcap u) \Rightarrow y) \sqcap \nu(\nu x \Rightarrow y) \\ &= (\mu x \Rightarrow \mu y) \sqcap (\nu x \Rightarrow \nu y) \end{aligned}$$

and, by T.3', (21), (22) and (18),

$$\begin{aligned}\mu(x \Rightarrow y) =& \mu(((x \sqcap u) \Rightarrow y) \sqcap (\nu x \Rightarrow y))\\ =& \mu((x \sqcap u) \Rightarrow y) \sqcap \mu(\nu x \Rightarrow y)\\ =& (\mu x \Rightarrow \mu(u \Rightarrow y)) \sqcap (\nu x \Rightarrow \mu y)\\ =& (\mu x \Rightarrow (\mu(u * u) \Rightarrow \mu y)) \sqcap (\nu x \Rightarrow \mu y).\end{aligned}$$

On the other hand, by (12),

$$\begin{aligned}x \Rightarrow y =& (\mu(x \Rightarrow y) \sqcap u) \sqcup \nu(x \Rightarrow y)\\ =& (\nu(x \Rightarrow y) \sqcup u) \sqcap \mu(x \Rightarrow y).\end{aligned}$$

In other words, $x \Rightarrow y$ is completely determined by $\mu(u * u)$ and the action of $\Rightarrow$ on $E(\mathcal{A})$.

- Using (I), the residuation principle, the first part of this proof and the definition of $\sqcap$ and $\sqcup$, we find the following equivalences, for all x, y and z in A:

$$\begin{cases}\nu(x * y) \leq \nu z\\ \mu(x * y) \leq \mu z\end{cases}$$

$$\text{iff} \quad x * y \leq z$$

$$\text{iff} \quad x \leq y \Rightarrow z$$

$$\text{iff} \quad \begin{cases}\nu x \leq \nu(y \Rightarrow z)\\ \mu x \leq \mu(y \Rightarrow z)\end{cases}$$

$$\text{iff} \quad \begin{cases}\nu x \leq (\nu y \Rightarrow \nu z) \sqcap (\mu y \Rightarrow \mu z)\\ \mu x \leq (\nu y \Rightarrow \mu z) \sqcap (\mu y \Rightarrow (\mu(u * u) \Rightarrow \mu z))\end{cases}$$

$$\text{iff} \quad \begin{cases}\nu x \leq \nu y \Rightarrow \nu z\\ \nu x \leq \mu y \Rightarrow \mu z\\ \mu x \leq \nu y \Rightarrow \mu z\\ \mu x \leq \mu y \Rightarrow (\mu(u * u) \Rightarrow \mu z)\end{cases}$$

$$\text{iff} \quad \begin{cases}\nu x * \nu y \leq \nu z\\ \nu x * \mu y \leq \mu z\\ \mu x * \nu y \leq \mu z\\ \mu x * \mu y * \mu(u * u) \leq \mu z\end{cases}$$

$$\text{iff} \quad \begin{cases}\nu x * \nu y \leq \nu z\\ (\nu x * \mu y) \sqcup (\mu x * \nu y) \sqcup (\mu x * \mu y * \mu(u * u)) \leq \mu z\end{cases}$$

In this equivalence, we can take $z = x * y$. Then $\nu(x * y) \leq \nu z$ and $\mu(x*y) \leq \mu z$ are obviously both satisfied. So we can conclude that $(\nu x*\mu y)\sqcup(\mu x*\nu y)\sqcup(\mu x*\mu y*\mu(u*u)) \leq \mu(x*y)$. On the other hand, we can also take $z = (\nu x*\mu y)\sqcup(\mu x*\nu y)\sqcup(\mu x*\mu y*\mu(u*u))$. In this case, one can easily verify that $\nu x*\nu y \leq \nu z$ and $(\nu x*\mu y)\sqcup(\mu x*\nu y)\sqcup(\mu x*\mu y*\mu(u*u)) \leq \mu z$ are both satisfied. So we find that $\mu(x*y) \leq (\nu x*\mu y)\sqcup(\mu x*\nu y)\sqcup(\mu x*\mu y*\mu(u*u))$. So $\mu(x*y) = (\nu x*\mu y)\sqcup(\mu x*\nu y)\sqcup(\mu x*\mu y*\mu(u*u))$. Together with (13) and (12), this completes the proof.

□

Using the one-to-one correspondence between IVRLs and triangle algebras [25], we can translate Theorem 14 to

Theorem 15 *Let* $(Int(\mathcal{L}), \sqcap, \sqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ *be an IVRL and* $\alpha \in L$, $* : L^2 \to L$ *and* $\Rightarrow: L^2 \to L$ *be defined by* $\alpha = pr_2([0,1] \odot [0,1])$, $x * y = pr_1([x,x] \odot [y,y])$ *and* $x \Rightarrow y = pr_1([x,x] \Rightarrow_\odot [y,y])$, *for all* x *and* y *in* L. *Then*

$$[x_1, x_2] \Rightarrow_\odot [y_1, y_2] = [(x_1 \Rightarrow y_1) \sqcap (x_2 \Rightarrow y_2), (x_1 \Rightarrow y_2) \sqcap (x_2 \Rightarrow (\alpha \Rightarrow y_2))]$$

and

$$[x_1, x_2] \odot [y_1, y_2] = [x_1 * y_1, (x_2 * y_2 * \alpha) \sqcup (x_1 * y_2) \sqcup (x_2 * y_1)],$$

for all $[x_1, x_2]$ *and* $[y_1, y_2]$ *in* $Int(\mathcal{L})$.

The negation is thus given by $\neg[x_1, x_2] = [x_1, x_2] \Rightarrow_\odot [0,0] = [(x_1 \Rightarrow 0)\sqcap(x_2 \Rightarrow 0), (x_1 \Rightarrow 0) \sqcap (x_2 \Rightarrow (\alpha \Rightarrow 0))] = [\neg x_2, \neg x_1 \sqcap \neg(x_2 * \alpha)]$.

In the following section, we will explore some consequences of this important theorem in the particular setting of $\mathcal{L}^I$. We conclude this section by listing a number of corollaries of Theorem 14, and a sufficient and necessary characterization of divisible IVRLs based on Theorem 15.

Corollary 16 *The negation* $\neg$ *in a triangle algebra* $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ *is an involution if and only if* $u * u = 0$ *and* $\neg\neg\nu x = \nu x$, *for all* x *in* A.

PROOF. Remark that $u * u = 0$ iff $\mu(u * u) = 0$. From the definition of the negation $\neg$, Theorem 14 and (8), it follows that the negation is determined by $\nu\neg x = \neg\mu x$ and $\mu\neg x = \neg(\mu x * \mu(u * u)) \sqcap \neg\nu x$.
Suppose that the negation is an involution. Then obviously $\neg\neg\nu x = \nu x$ for all

x in A. Furthermore, we have $0 = \nu u = \nu\neg\neg u = \neg\mu\neg u = \neg\neg\mu(u*u) = \mu(u*u)$. Conversely, if $u * u = 0$ and $\neg\neg\nu x = \nu x$, for all x in A, then $\mu\neg x = \neg\nu x$. So $\nu\neg\neg x = \neg\mu\neg x = \neg\neg\nu x = \nu x$ and $\mu\neg\neg x = \neg\nu\neg x = \neg\neg\mu x = \neg\neg\nu\mu x = \nu\mu x = \mu x$, which implies $\neg\neg x = x$. □

Remark that this property is a generalization of Proposition 4.4 in [23], which stated the result for IVRLs on $\mathcal{L}^I$.

Corollary 17 *In a triangle algebra $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, the following inequalities and identities hold, for every x and y in A:*

$$\mu x \Rightarrow \nu y \leq \nu(x \Rightarrow y) \tag{23}$$

$$\nu x \leq \mu(u * x) \tag{24}$$

$$(\nu x \Rightarrow \nu y) * (\mu x \Rightarrow \mu y) \leq x \Rightarrow y \tag{25}$$

$$x \Rightarrow (y \sqcup u) = (\nu x \Rightarrow \nu y) \sqcup u \tag{26}$$

$$\mu(x \Rightarrow (y \sqcap u)) = \mu(x \Rightarrow y) = \mu(x \Rightarrow \mu y) \tag{27}$$

PROOF.

(23) Because of T.1, T.1' and (M), $\mu x \Rightarrow \nu y \leq (\nu x \Rightarrow \nu y) \sqcap (\mu x \Rightarrow \mu y) = \nu(x \Rightarrow y)$.

(24) Using T.5', we find $\nu x \leq (\nu u * \mu x) \sqcup \nu x \sqcup (\mu(u * u) * \mu u * \mu x) = (\nu u * \mu x) \sqcup (\mu u * \nu x) \sqcup (\mu(u * u) * \mu u * \mu x) = \mu(u * x)$.

(25) By (1) and T.1, $(\nu x \Rightarrow \nu y) * (\mu x \Rightarrow \mu y) \leq (\nu x \Rightarrow \nu y) \sqcap (\mu x \Rightarrow \mu y) = \nu(x \Rightarrow y) \leq x \Rightarrow y$.

(26) On the one hand, we have $\mu(x \Rightarrow (y \sqcup u)) = 1 = \mu((\nu x \Rightarrow \nu y) \sqcup u)$ because of (2), (M) and T.5'. On the other hand, $\nu(x \Rightarrow (y \sqcup u)) = (\nu x \Rightarrow \nu(y \sqcup u)) \sqcap (\mu x \Rightarrow \mu(y \sqcup u)) = (\nu x \Rightarrow \nu y) \sqcap (\mu x \Rightarrow 1) = \nu x \Rightarrow \nu y = \nu((\nu x \Rightarrow \nu y) \sqcup u)$, by T.4, T.5, T.5', (M) and (C). So by T.8, $x \Rightarrow (y \sqcup u) = (\nu x \Rightarrow \nu y) \sqcup u$.

(27) By T.1' and T.2', we have $\mu(x \Rightarrow y) = (\mu x \Rightarrow (\mu(u * u) \Rightarrow \mu y)) \sqcap (\nu x \Rightarrow \mu y) = (\mu x \Rightarrow (\mu(u * u) \Rightarrow \mu\mu y)) \sqcap (\nu x \Rightarrow \mu\mu y) = \mu(x \Rightarrow \mu y)$. The proof of the other identity is analogous, using T.3' and T.5'.

□

Proposition 18 *Let $(Int(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ be an IVRL and $\alpha \in L$, $* : L^2 \to L$ and $\Rightarrow : L^2 \to L$ be defined by $\alpha = pr_2([0,1] \odot [0,1])$, $x * y = pr_1([x,x] \odot [y,y])$ and $x \Rightarrow y = pr_1([x,x] \Rightarrow_\odot [y,y])$, for all x and y in L. Then this IVRL is divisible if, and only if, $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a Heyting-algebra and $\alpha \sqcup x \sqcup \neg x = 1$ for all x in L.*

PROOF. Suppose $(Int(\mathcal{L}), \sqcap, \sqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ is a divisible IVRL. Now consider, for any fixed x in L, the intervals $[x,x]$ and $[x,1]$. Because of the divisibility, there must exist an interval $[y,z]$ such that $[x,x] = [x,1] \odot [y,z] = [x*y, (x*z) \sqcup (1*y) \sqcup (1*z*\alpha)]$. So $x = x*y$ and $y \leq x$, which implies $x*x = x$ (no matter what the value of α is). As this holds for any x, we find, for any y and z, that $y \sqcap z = (y \sqcap z) * (y \sqcap z) \leq y * z \leq y \sqcap z$, so $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a Heyting-algebra. Furthermore, we have $[0,1] = [x,1] \sqcap [0,1] = [x,1] \odot ([x,1] \Rightarrow_\odot [0,1]) = [x,1] \odot [\neg x, 1] = [\neg x * x, x \sqcup \neg x \sqcup \alpha]$.
Conversely, suppose $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a Heyting-algebra (and therefore divisible and distributive) and $\alpha \sqcup x \sqcup \neg x = 1$ for all x in L. Then $\sqcap$ and $*$ coincide. We prove that $[x_1,x_2] \sqcap [y_1,y_2] = [x_1,x_2] \odot ([x_1,x_2] \Rightarrow_\odot [y_1,y_2])$, or in other words: that $x_1 \sqcap y_1 = x_1 \sqcap (x_1 \Rightarrow y_1) \sqcap (x_2 \Rightarrow y_2)$ and that $x_2 \sqcap y_2$ is the supremum of $\alpha \sqcap x_2 \sqcap ((x_2 \sqcap \alpha) \Rightarrow y_2) \sqcap (x_1 \Rightarrow y_2)$, $x_2 \sqcap (x_1 \Rightarrow y_1) \sqcap (x_2 \Rightarrow y_2)$ and $x_1 \sqcap ((x_2 \sqcap \alpha) \Rightarrow y_2) \sqcap (x_1 \Rightarrow y_2)$, for every $[x_1,x_2]$ and $[y_1,y_2]$ in $Int(\mathcal{L})$. Indeed, $x_1 \sqcap (x_1 \Rightarrow y_1) \sqcap (x_2 \Rightarrow y_2) = x_1 \sqcap y_1 \sqcap (x_2 \Rightarrow y_2) = x_1 \sqcap y_1$ (because $y_1 \leq y_2 \leq x_2 \Rightarrow y_2$), and the supremum of

- $\alpha \sqcap x_2 \sqcap ((x_2 \sqcap \alpha) \Rightarrow y_2) \sqcap (x_1 \Rightarrow y_2) = \alpha \sqcap x_2 \sqcap y_2 \sqcap (x_1 \Rightarrow y_2) = \alpha \sqcap x_2 \sqcap y_2$,
- $x_2 \sqcap (x_1 \Rightarrow y_1) \sqcap (x_2 \Rightarrow y_2) = (x_1 \Rightarrow y_1) \sqcap x_2 \sqcap y_2$ and
- $x_1 \sqcap ((x_2 \sqcap \alpha) \Rightarrow y_2) \sqcap (x_1 \Rightarrow y_2) = x_1 \sqcap y_2 \sqcap ((x_2 \sqcap \alpha) \Rightarrow y_2) = x_1 \sqcap y_2 = x_1 \sqcap x_2 \sqcap y_2$

is, by distributivity, $(x_2 \sqcap y_2) \sqcap (\alpha \sqcup (x_1 \Rightarrow y_1) \sqcup x_1) = x_2 \sqcap y_2$ (because $1 = \alpha \sqcup \neg x_1 \sqcup x_1 \leq \alpha \sqcup (x_1 \Rightarrow y_1) \sqcup x_1$). □

We can distinguish two special cases in the previous proposition:

- If $\alpha = 1$ (the product $\odot$ is t-representable), then the condition $\alpha \sqcup x \sqcup \neg x = 1$ is always fulfilled. So, every Heyting algebra on $\mathcal{L}$ corresponds with a Heyting algebra on $\mathbb{T}(\mathcal{L})$ and vice versa. Indeed: the t-representable extension of the infimum on $\mathcal{L}$, defined by, for $x = [x_1,x_2]$ and $y = [y_1,y_2]$ in $Int(\mathcal{L})$,

$$\mathcal{T}_{\sqcap,1}(x,y) = [x_1 \sqcap y_1, x_2 \sqcap y_2], \tag{28}$$

  is equal to the infimum on $\mathbb{T}(\mathcal{L})$.
- If $\alpha = 0$ (the product $\odot$ is pseudo t-representable), then the condition $\alpha \sqcup x \sqcup \neg x = 1$ becomes $x \sqcup \neg x = 1$. The only Heyting algebras in which this holds for any x, are Boolean algebras. In this case, the IVRL is not only divisible, but even an MV-algebra. This was already proven in [16].

Remark that the condition $\alpha \sqcup x \sqcup \neg x = 1$, for any x in L, implies that $\alpha \sqcup \neg\alpha = 1$. If $\alpha \sqcup \neg\alpha = 1$, one does not need to verify the condition for $x \leq \alpha$ nor for $\neg\alpha \leq x$. Indeed, if $x \leq \alpha$, then $1 = \alpha \sqcup \neg\alpha \leq \alpha \sqcup x \sqcup \neg\alpha \leq \alpha \sqcup x \sqcup \neg x$; if $\neg\alpha \leq x$, then $1 = \alpha \sqcup \neg\alpha \leq \alpha \sqcup \neg\alpha \sqcup \neg x \leq \alpha \sqcup x \sqcup \neg x$.

4 Standard IVRLs

In this section, we interpret the results that we have established in the previous section for the specific case of IVRLs on $\mathcal{L}^I$.

First remark that the base lattice $([0,1], \min, \max)$ is linear, and therefore every IVRL (or, equivalently, triangle algebra) on $\mathcal{L}^I$ is pseudo-linear. However, none of them is prelinear, since, as was shown in [6], there does not exist an MTL-algebra on $\mathcal{L}^I$.

In Example 7, we have seen that each t-norm $\mathcal{T}_{T,\alpha}$ on $\mathcal{L}^I$ defined by Formula (9) generates an IVRL. According to Theorem 15, the converse of this result holds as well: if $(\mathcal{L}^I, \sqcap, \sqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ is an IVRL, then it is generated by a t-norm of this class. Indeed, if T is defined by $T(x,y) = pr_1([x,x] \odot [y,y])$ and α by $\alpha = pr_2([0,1] \odot [0,1])$, this theorem implies

$$[x_1, x_2] \odot [y_1, y_2] = [T(x_1, y_1), \max(T(\alpha, T(x_2, y_2)), T(x_1, y_2), T(x_2, y_1))],$$

which corresponds exactly to Formula (9).

This result enables us to take a crucial step forward in solving a long-standing open problem, raised for the first time in [7], regarding the characterization of t-norms on $\mathcal{L}^I$ satisfying the residuation principle. The case of t-representable t-norms has already been settled in [8], but for other t-norms so far only sufficient conditions for satisfying the residuation principle had been obtained ([7,9]). This emphasizes the relevance of Theorem 15.

On the other hand, it should be noted that Theorem 15 does not consider t-norms and implicators under which the diagonal of $\mathcal{L}^I$ is not closed. Such operations do exist [3], see e.g. [25]. While their practical use can be debated, from a mathematical point of view, it remains a challenge to extend our characterization to include these RLs on triangularizations as well. It is doubtful, however, whether this more general class can also be captured by a variety like triangle algebras.

As every MTL-algebra on the unit interval is induced by a left-continuous t-norm, we can conclude that every IVRL on $\mathcal{L}^I$ corresponds in a unique way to a couple (T, α), in which T is a left-continuous t-norm on $([0,1], \min, \max)$ and $\alpha \in [0,1]$. If we take a continuous t-norm on the diagonal of $\mathcal{L}^I$ instead of a left-continuous one, then the structure on the diagonal is a BL-algebra instead of an MTL-algebra. In other words, in this case the IVRL is pseudo-divisible. So a lot of pseudo-divisible IVRLs exist. Surprisingly, because of

[3] Interestingly, so far no RL on a triangularization has been found such that the diagonal is closed under neither the product nor the implication.

Proposition 18, there is only one divisible IVRL on $\mathcal{L}^I$, namely the one induced by $\mathcal{T}_{\min,1}$. Indeed: there is only one Heyting algebra on $([0,1], \min, \max)$; and if $\alpha \sqcup x \sqcup \neg x = 1$ for every x in $[0,1]$, then necessarily $\alpha = 1$ (because $x \sqcup \neg x$ is not equal to 1 for x in $]0,1[$, as $\neg x = 1$ iff $x = 0$).

On the other hand, an IVRL on $\mathcal{L}^I$ (or in fact, any IVRL) can only have an involutive negation if $\alpha = 0$, because of Proposition 16. Therefore, no divisible IVRL with involutive negation (i.e. no MV-algebra) exists on $\mathcal{L}^I$. This is in accordance with the result that no MTL-algebra exists on $\mathcal{L}^I$. These observations once more illustrate the different characteristics of t-representable and pseudo t-representable t-norms on $\mathcal{L}^I$.

5 Conclusion and Future Work

In this paper, we have established an important result about triangle algebras, and thus also about IVRLs: the product in triangle algebras is necessarily an extension of the product on a smaller residuated lattice. We have shown that these extensions are parametrized by the elements of the smaller residuated lattice and have derived the specific form of these extensions. We have used this result to study other important properties of IVRLs, and to find necessary and sufficient conditions for IVRLs to satisfy these properties. Furthermore, we have considered in some more detail the case of $\mathcal{L}^I$, the most commonly used lattice for building IVRLs.

It is known that MTL-algebras are subdirect products of linear residuated lattices [12]. An important challenge in our current research is to prove a similar result for triangle algebras, i.e., to find out which subclass of triangle algebras contains the subdirect products of triangle algebras with a linear diagonal. In the same context, we would also like to unravel all connections between the different properties that can be imposed on triangle algebras.

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