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ON THE PROPERTIES OF A GENERALIZED CLASS OF T-NORMS IN INTERVAL-VALUED FUZZY LOGICS

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Since it does not generate any MTL-algebra (prelinear residuated lattice), the lattice $\mathcal{L}^I$ of closed subintervals of $[0,1]$ falls outside the mainstream of research on formal fuzzy logics. However, due to the intimate connection between logical connectives on $\mathcal{L}^I$ and those on $[0,1]$, many relevant logical properties can still be maintained, sometimes in a slightly weaker form. In this paper, we focus on a broad class of parametrized t-norms on $\mathcal{L}^I$. We derive their corresponding residual implicators, and examine commonly imposed logical properties. Importantly, we formally establish one-to-one correspondences between $\vee$-definability (respectively, weak divisibility) for t-norms of this class and strong $\vee$-definability (resp., divisibility) for their counterparts on $[0,1]$.

Keywords: Formal fuzzy logic; interval-valued fuzzy sets; triangular norms; residuated lattices.

1. Introduction

Formal fuzzy logic (also: fuzzy logic in the narrow sense) is concerned with the development of many-valued logics that offer a graded approach to vagueness, and has drawn considerable attention over the past decade (see e.g. the monographs in Refs. 1–3). In principle, the evaluation set (set of truth values) of a formal fuzzy logic may be any bounded lattice; typically, however the focus is on t-norm based residuated fuzzy logics, which are $[0,1]$-valued residuated propositional systems in which a t-norm and its residuum are chosen as truth functions for conjunction and implication. They can be placed in a hierarchy of logics according to their characteristic axioms,[4] and their algebraic counterparts are special subvarieties of residuated lattices, defined by particular equations; for example, MTL (Monoidal T-norm based Logic) corresponds to the variety of prelinear residuated lattices (also called MTL-algebras), while Hajek's BL (Basic Logic) generates the variety of divisible MTL-algebras (BL-algebra's). On $[0,1]$, there is a one-to-one correspondence between left-continuous t-norms and MTL-algebras, and between continuous t-norms and BL-algebras, respectively. Higher up the hierarchy, Łukasiewicz, product and Gödel logics correspond to the main continuous t-norms: Łukasiewicz, product and minimum, respectively.

On the other hand, $[0,1]$-valued systems are limited in their expressivity because they can only model vagueness (gradations in the notion of truth), while recently there has been a rising interest in logical formalisms[5] that can deal also with a feature of uncertainty (lack of information). In particular, interval-valued fuzzy logics are encountered in a growing number of applications.[6] The main reasons for their success are:

- the robust underlying mathematical framework: interval-valued fuzzy sets may be seen as a special kind of $\mathcal{L}$-fuzzy sets in the sense of Goguen[7] w.r.t. the lattice $\mathcal{L}^I$ of closed subintervals of $[0,1]$;
- the straightforward interpretation of the truth degrees: intervals may be understood to approximate the exact, but incompletely known, truth degree; and
- the limited computational complexity.

While many t-norms, parallelling and extending the main choices on the unit interval, can be chosen on $\mathcal{L}^I$ to make it a residuated lattice, no t-norm on $\mathcal{L}^I$ induces an MTL-algebra. In this paper, by investigating the properties of t-norms from the broad parametrized class introduced by Deschrijver and Kerre,[8] we hope to learn more about this structure.

The remainder of this paper is structured as follows. In Sec. 2, we recall basic notions about interval-valued fuzzy set theory, t-norms and varieties of residuated lattices. In Sec. 3, we derive the residual implicators corresponding to Deschrijver and Kerre's t-norms. Then, Sec. 4 examines how particular properties that define certain generalizations of Boole-algebras are transferred between $[0,1]$ and $\mathcal{L}^I$.

2. Preliminaries

Definition 2.1.[9] We define $\mathcal{L}^I = (L^I, \wedge, \vee)$, where

$$L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0,1]^2 \text{ and } x_1 \leq x_2\},$$

and for all $[x_1, x_2]$ and $[y_1, y_2]$ in L^I,

$$[x_1, x_2] \wedge [y_1, y_2] = [\min(x_1, y_1), \min(x_2, y_2)],$$
$$[x_1, x_2] \vee [y_1, y_2] = [\max(x_1, y_1), \max(x_2, y_2)].$$

Furthermore we define the set $D = \{[x, x] \mid x \in [0,1]\} \subset L^I$. From now on, if $x \in L^I$, we denote its lower bound by x_1 and its upper bound by x_2, i.e. $x = [x_1, x_2]$.

The structure $\mathcal{L}^I$ is a complete (and thus bounded) lattice. The corresponding partial order $\leq$ is given by: $[x_1, x_2] \leq [y_1, y_2]$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$. In this lattice, the smallest element $[0,0]$ is meet-irreducible and the largest element $[1,1]$

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