Filters of Interval-Valued Residuated Lattices

Bart Van Gasse, Chris Cornelis, Glad Deschrijver, and Etienne Kerre

Fuzziness and Uncertainty Modeling Research Unit,
Department of Applied Mathematics and Computer Science, Ghent University,
Krijgslaan 281 (S9), B-9000 Gent, Belgium
{Bart.VanGasse,Chris.Cornelis,Glad.Deschrijver,Etienne.Kerre}@UGent.be

Abstract. An important concept in the theory of residuated lattices and other algebraical structures used for fuzzy formal logic, is that of a filter. Filters can be used, amongst others, to define congruence relations. Specific kinds of filters include Boolean filters and prime filters.
In this paper, we define several different filters of interval-valued residuated lattices and examine their mutual dependencies and connections. We also show that these filters are determined by their intersection with the set of exact elements, i.e., the intervals consisting of precisely one element.

Key words: filters, residuated lattices, interval-valued structures

1 Introduction and Preliminaries

In this paper, a residuated lattice is defined as a structure $\mathcal{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ in which $\sqcap, \sqcup, *$ and $\Rightarrow$ are binary operators on L and

- $(L, \sqcap, \sqcup)$ is a bounded lattice with 0 as smallest and 1 as greatest element,
- $*$ is commutative and associative, with 1 as neutral element, and
- $x * y \leq z$ iff $x \leq (y \Rightarrow z)$ for all x, y and z in L (residuation principle).

The (possibly partial) ordering $\leq$ and negation $\neg$ in a residuated lattice $\mathcal{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ are defined as follows, for all x and y in L: $x \leq y$ iff $x \sqcap y = x$ (or equivalently, iff $x \sqcup y = y$; or, also equivalently, iff $x \Rightarrow y = 1$) and $\neg x = x \Rightarrow 0$.
Residuated lattices [1] form a variety and constitute the semantics of Höhle's Monoidal Logic (ML) [7], which is the basis for the majority of formal fuzzy logics, like Esteva and Godo's Monoidal T-norm based Logic (MTL) [3], Hájek's Basic Logic (BL) [5], Łukasiewicz Logic (LL) [9], Intuitionistic Logic (IL) [6] and Gödel Logic (GL) [2,4].

An MTL-algebra [3] is a prelinear residuated lattice, i.e., a residuated lattice in which $(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$ for all x and y in L. Linear residuated lattices are always prelinear. In linear residuated lattices, 1 is always $\sqcup$-irreducible. This means that for all x and y in L, $x \sqcup y = 1$ iff $x = 1$ or $y = 1$ (or both).
A Boole-algebra [8] is a residuated lattice in which $x \sqcup \neg x = 1$ for all x in L. It is always prelinear.

Definition 1. *[5,7] A filter of a residuated lattice $\mathcal{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a non-empty subset F satisfying the conditions*

- *for all x and y in L: if $x \in F$ and $x \leq y$, then $y \in F$,*
- *for all x and y in F: $x * y \in F$ (i.e., F is closed under $*$).*

A Boolean filter of $\mathcal{L}$ is a filter F satisfying

- *for all x in L: $x \sqcup \neg x \in F$.*

A prime filter of $\mathcal{L}$ is a filter F satisfying

- *for all x and y in L: $x \Rightarrow y \in F$ or $y \Rightarrow x \in F$ (or both).*

A prime filter of the second kind is a filter F satisfying

- *for all x and y in L: if $x \sqcup y \in F$, then $x \in F$ or $y \in F$ (or both).*

It can easily be seen that L is an instance of each of these four kinds of filters, and that every filter always contains the greatest element 1. The singleton $\{1\}$ is always a filter, but it is only a Boolean filter if $\mathcal{L}$ is a Boole-algebra (in this case, all filters are Boolean filters). It is only a prime filter if $\mathcal{L}$ is linearly ordered (in this case, all filters are prime filters), and only a prime filter of the second kind if 1 is $\sqcup$-irreducible in $\mathcal{L}$.
Given a filter F of a residuated lattice $\mathcal{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$, we can define the relation $\sim_F$ as follows: for all x and y in L, $x \sim_F y$ iff $x \Rightarrow y \in F$ and $y \Rightarrow x \in F$. This relation is a congruence on $\mathcal{L}$.

Proposition 1. *Let $\mathcal{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ be a residuated lattice. Every prime filter of $\mathcal{L}$ is also a prime filter of the second kind.*
If $\mathcal{L}$ is an MTL-algebra, then every prime filter of the second kind of $\mathcal{L}$ is also a prime filter.

In residuated lattices that are not MTL-algebras, prime filters of the second kind are in general not prime filters. As a counterexample we can take a residuated lattice on the lattice in Figure 1. As 1 is $\sqcup$-irreducible, $\{1\}$ will be a prime filter of the second kind. But not a prime filter, as $a \Rightarrow b \neq 1$ and $b \Rightarrow a \neq 1$.

In residuated lattices (even if they are prelinear), not all prime filters are Boolean filters, and vice versa. As counterexamples, we can consider the one-element filters of a residuated lattice on a three-element chain, and of the Boole-algebra on the powerset of a two-element set, respectively.

Now we recall our definition of IVRL from [10].

Definition 2. Given a lattice $\mathcal{L} = (L, \sqcap, \sqcup)$, its triangularization $\mathbb{T}(\mathcal{L})$ is the structure $\mathbb{T}(\mathcal{L}) = (Int(\mathcal{L}), \bigsqcap, \bigsqcup)$ defined by

- $Int(\mathcal{L}) = \{[x_1, x_2] \mid (x_1, x_2) \in L^2 \text{ and } x_1 \leq x_2\}$,
- $[x_1, x_2] \bigsqcap [y_1, y_2] = [x_1 \sqcap y_1, x_2 \sqcap y_2]$,
- $[x_1, x_2] \bigsqcup [y_1, y_2] = [x_1 \sqcup y_1, x_2 \sqcup y_2]$.

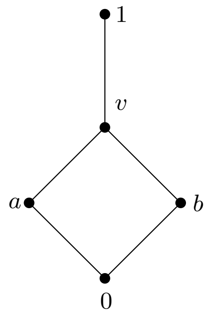

Fig. 1. The singleton $\{1\}$ is a prime filter of the second kind, but not a prime filter.

The elements of the set $D_{\mathcal{L}} = \{[x, x] \mid x \in L\}$ are called the exact elements of $\mathbb{T}(\mathcal{L})$.
The first and the second projection pr_1 and pr_2 are the mappings from $T(\mathcal{L})$ to L, defined by $pr_1([x_1, x_2]) = x_1$ and $pr_2([x_1, x_2]) = x_2$, for all $[x_1, x_2]$ in $T(\mathcal{L})$.

The triangularization of $([0, 1], \min, \max)$ is denoted as $\mathcal{L}^I = (L^I, \sqcap, \sqcup)$.

Definition 3. An interval-valued residuated lattice (IVRL) is a residuated lattice $(Int(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0, 0], [1, 1])$ on the triangularization $\mathbb{T}(\mathcal{L})$ of a bounded lattice $\mathcal{L}$, in which $D_{\mathcal{L}}$ is closed under $\odot$ and $\Rightarrow_\odot$, i.e., $[x, x] \odot [y, y] \in D_{\mathcal{L}}$ and $[x, x] \Rightarrow_\odot [y, y] \in D_{\mathcal{L}}$ for all x, y in L.

The set of exact elements of an IVRL contains $[0, 0]$ and $[1, 1]$ and is closed under $\bigsqcap$, $\bigsqcup$, $\odot$ and $\Rightarrow_\odot$. Thus it forms a subalgebra of the IVRL.
In [11], we proved

Theorem 1. *Let $(Int(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0, 0], [1, 1])$ be an IVRL and $\alpha \in L$, $* : L^2 \to L$ and $\Rightarrow: L^2 \to L$ be defined by $\alpha = pr_2([0, 1] \odot [0, 1])$, $x * y = pr_1([x, x] \odot [y, y])$ and $x \Rightarrow y = pr_1([x, x] \Rightarrow_\odot [y, y])$, for all x and y in L. Then*

$$[x_1, x_2] \Rightarrow_\odot [y_1, y_2] = [(x_1 \Rightarrow y_1) \sqcap (x_2 \Rightarrow y_2), (x_1 \Rightarrow y_2) \sqcap (x_2 \Rightarrow (\alpha \Rightarrow y_2))]$$

and

$$[x_1, x_2] \odot [y_1, y_2] = [x_1 * y_1, (x_2 * y_2 * \alpha) \sqcup (x_1 * y_2) \sqcup (x_2 * y_1)],$$

for all $[x_1, x_2]$ and $[y_1, y_2]$ in $Int(\mathcal{L})$.

2 Filters on interval-valued residuated lattices

As IVRLs are special kinds of residuated lattices, the above mentioned definitions for filters can be used for IVRLs as well. The quotient algebras determined

by these filters are residuated lattices. However, they will not always be interval-valued residuated lattices. As an example, we take the smallest possible triangularization (except for the trivial one), with three elements ($[0,0]$, $[0,1]$ and $[1,1]$) and fix an IVRL on it by choosing $[0,1] \odot [0,1] = [0,1]$. Then the subset $\{[0,1],[1,1]\}$ is a filter (even a prime filter), but the corresponding quotient algebra has exactly two elements: $\{[0,0]\}$ and $\{[0,1],[1,1]\}$. Therefore it cannot be (isomorphic to) an IVRL.
To ensure that the quotient algebras will be IVRLs, we require an extra condition in the definition of a filter:

Definition 4. *[12] Let $(Int(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ be an IVRL. An IVRL-filter of $(Int(\mathcal{L}), \bigsqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0,0], [1,1])$ is a non-empty subset F satisfying*

- *for all $[x_1,x_2]$ and $[y_1,y_2]$ in L: if $[x_1,x_2] \in F$ and $[x_1,x_2] \leq [y_1,y_2]$, then $[y_1,y_2] \in F$,*
- *for all $[x_1,x_2]$ and $[y_1,y_2]$ in F: $[x_1,x_2] \odot [y_1,y_2] \in F$ (i.e., F is closed under $\odot$),*
- *for all $[x_1,x_2]$ in F: $[x_1,x_1] \in F$ (i.e., F is closed under pr_1).*

Remark that because of the first condition, F is also closed under pr_2. Moreover, together with the third condition the first condition implies $[x_1,x_2] \in F$ iff $[x_1,x_1] \in F$. Thus an IVRL-filter is completely determined by its intersection with the subset of exact elements. This immediately suggests two ways for defining different kinds of IVRL-filters: on the one hand, we can define filters with conditions that need to hold for all elements; on the other hand, we can define filters on the subalgebra of exact elements (a residuated lattice) and extend this definition to all elements by stating $[x_1,x_2] \in F$ iff $[x_1,x_1] \in F$.

Definition 5. *A Boolean IVRL-filter (prime IVRL-filter, prime IVRL-filter of the second kind, respectively) is defined as a Boolean filter (prime filter, prime filter of the second kind, respectively) (of the IVRL as a residuated lattice), but with the extra requirement that it should be closed under pr_1.*
An IVRL-extended Boolean filter (IVRL-extended prime filter, IVRL-extended prime filter of the second kind, respectively) is defined as a subset (of the IVRL) whose exact elements form a Boolean filter (a prime filter, a prime filter of the second kind, respectively) of the residuated lattice of all exact elements, with the requirement that it contains an interval $[x_1,x_2]$ iff it contains the exact element $[x_1,x_1]$.

Obviously, every Boolean IVRL-filter (prime IVRL-filter, prime IVRL-filter of the second kind, respectively) is also an IVRL-extended Boolean filter (IVRL-extended prime filter, IVRL-extended prime filter of the second kind, respectively). We now summarize the other connections between these definitions.

Proposition 2.

- *Every IVRL-extended prime filter of the second kind is also a prime IVRL-filter of the second kind.*
- *The only Boolean IVRL-filter is the trivial one (the whole IVRL).*

– *Every IVRL-extended prime filter is also an IVRL-extended prime filter of the second kind. If the exact elements of the IVRL form a prelinear residuated lattice, the converse holds as well.*
– *Prime IVRL-filters are exactly IVRL-extended prime filters that are at the same time IVRL-extended Boolean filters.*

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