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Sinha–Dougherty approach to the fuzzification of set inclusion revisited

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Abstract

Inclusion for fuzzy sets was first introduced by Zadeh in his seminal 1965 paper. Since it was found that the definition of inclusion was not in the true spirit of fuzzy logic, various researchers have set out to define alternative indicators of the inclusion of one fuzzy set into another. Among these alternatives, the indicator proposed by Sinha and Dougherty stands out as an intuitively appealing one, as it is built up with a strong but appropriate collection of axioms in mind. Starting from a very general expression depending on four functional parameters for such an indicator, those authors proposed conditions they claimed to be necessary and sufficient to satisfy the axioms. This paper aims to revisit this material by exposing it in a clearer way, correcting errors along the way while pinpointing some nasty pitfalls that Sinha and Dougherty overlooked. This results in a new, easier to handle and more consistent framework for the axiomatic characterization of inclusion grades for fuzzy sets, advantageous to the further development of practical applications. In the end, a link is established with Kitainik's results on the fuzzification of set inclusion, allowing amongst others the derivation of a sufficient and necessary characterization of the Sinha–Dougherty axioms. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Traditionally, fuzzy set inclusion is defined according to Zadeh's [13] original proposal. For A and B fuzzy sets in a universe X he defined: $A \subseteq B \Leftrightarrow (\forall x \in X)(A(x) \leqslant B(x))$, i.e. $A \subseteq B$ if and only if the graph of A fits beneath the graph of B. This rigid definition unfortunately does not do justice to the spirit of fuzzy set theory: we may want to talk about a fuzzy set being "more or less" a subset of another one, and such a concept has been successfully applied in a wide range of domains like,

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e.g. mathematical morphology [2], approximate reasoning [3,4] and fuzzy entropy [9]. Very aptly, Bandler and Kohout [1] call the definition of inclusion by Zadeh "an unconscious step backwards in the realm of dichotomy".

This intuition has inspired several researchers to consider $\mathscr{F}(X)\times\mathscr{F}(X)\to[0,1]$ mappings *Inc* (where $\mathscr{F}(X)$ represents the class of fuzzy sets in a universe X), such that the value $Inc(A,B)$ indicates to what extent A is included into B. Of course, we should impose some elementary constraints on this indicator in order to come up with a genuine implementation of inclusion. Two axiom systems seem to prevail in the literature, notably, the one proposed by Young [12] and the one by Sinha and Dougherty [11]. The main difference between them lies in how each treats inclusion for crisp sets. While the Sinha–Dougherty axioms explicitly demand that $Inc(A,B)\in\{0,1\}$ for A and $B\in\mathscr{P}(X)$, where $\mathscr{P}(X)$ denotes the power class of X, Young disputes this condition and she consequently does not impose it, arguing that much of the relative structure of A and B is lost in this way. We give two arguments in favour of Sinha and Dougherty's line of reasoning:

1. By constraining *Inc* to only two values for crisp sets, the indicator of inclusion grade is a faithful (backwards compatible) extension of the concept of crisp inclusion.
2. Taking the following definition of inclusion in the crisp case:

$$A\subseteq B \;\Leftrightarrow\; (\forall x\in X)(x\in A\Rightarrow x\in B)$$

and using a direct fuzzification to implement *Inc*, as suggested by Bandler and Kohout [1], we arrive at:

$$Inc(A,B)=\inf_{x\in X}\mathscr{I}(A(x),B(x)),$$

where $\mathscr{I}$ is a fuzzy implicator [10], i.e. a $[0,1]^2\to[0,1]$-mapping for which $\mathscr{I}(0,0)=\mathscr{I}(0,1)=\mathscr{I}(1,1)=1$, $\mathscr{I}(1,0)=0$ and whose first (second) partial mappings are decreasing (increasing). It can be verified that such *Inc* is always two-valued for crisp sets.

This said, we now concentrate exclusively on the Sinha–Dougherty approach, S–D approach for short. In Section 2, we will briefly survey the proposed axioms, reducing their number from nine to eight by resolving some dependencies among them. At this point, the general Sinha–Dougherty indicator, a mathematical formula with four functional parameters, is also introduced.

Section 3 then offers a critical review of the original properties for these parameters. The objective of this discussion is basically threefold: first, we prove that the authors' claim for necessity and sufficiency of the properties to satisfy the axioms is unjustified; we will provide some convincing counterexamples. Secondly, we will derive supplementary conditions, resulting in a collection of (at least) sufficient requirements for the axioms, while at the same time allowing for a wider range of (meaningful) indicators than the original S–D properties did. Finally, we will indicate why the modified properties are still not necessary for the axioms; we will show in particular that the specific form of the indicator chosen in [11] gets much of the blame for this.

To alleviate the problems encountered with the general S–D indicator, we rewrite its definition in Section 4, and derive a much simpler set of sufficient conditions. In this new framework, checking for necessity becomes more elementary, allowing us to identify clearly the complicating factors in this process.

In Section 5, we look at an alternative approach to the classification of fuzzy inclusion indicators due to Kitainik. His results ultimately lead us to a necessary and sufficient characterization of the Sinha–Dougherty axioms. It turns out that indicators satisfying all axioms necessarily belong to a special subclass of the Bandler–Kohout indicator family. As will be seen, in case we are working with a finite universe X, the requirements on the fuzzy implicator $\mathscr{I}$ can be written down in an even more explicit way.

Section 6 concludes with some general remarks and options for future research.

2. Most general form of the Sinha–Dougherty indicator

We will start by giving a reduced list of axioms as proposed in [11].

Definition 2.1 (Sinha–Dougherty axioms). Let *Inc* be a $\mathscr{F}(X)\times\mathscr{F}(X)\to[0,1]$ mapping, and A, B and C fuzzy sets in a given universe X. The Sinha–Dougherty axioms imposed on *Inc* are as follows:

Axiom 1. $Inc(A,B)=1 \Leftrightarrow A\subseteq B$ (in Zadeh's sense).

Axiom 2. $Inc(A,B)=0 \Leftrightarrow ker(A)\cap co\,supp(B)\neq\emptyset$, where $ker(A)=\{x\in X|A(x)=1\}$ and $supp(B)=\{x\in X|B(x)>0\}$.

Axiom 3. $B\subseteq C\Rightarrow Inc(A,B)\leqslant Inc(A,C)$, i.e. the indicator has increasing second partial mappings.

Axiom 4. $B\subseteq C\Rightarrow Inc(C,A)\leqslant Inc(B,A)$, i.e. the indicator has decreasing first partial mappings.

Axiom 5. $Inc(A,B)=Inc(S(A),S(B))$ where S is a $\mathscr{F}(X)\to\mathscr{F}(X)$ mapping defined by, for every $x\in X$, $S(A)(x)=A(s(x))$, s denoting an $X\to X$ mapping.

Axiom 6. $Inc(A,B)=Inc(co\,B,co\,A)$.

Axiom 7. $Inc(B\cup C,A)=\min(Inc(B,A),Inc(C,A))$.

Axiom 8. $Inc(A,B\cap C)=\min(Inc(A,B),Inc(A,C))$.

The original version included a ninth axiom, $Inc(A,B\cup C)\geqslant\max(Inc(A,B),Inc(A,C))$. Frago [6] indicated that it is redundant because, as can easily be verified, it is equivalent to axiom 3.

Next, we recall the most general form for the indicator similar [1] to the formula appearing in [11]. The following mappings are introduced:

- $\theta:\mathscr{P}([0,1])\to[0,1]$, e.g. supremum, infimum

[1] In Sinha and Dougherty's proposal, θ acts on multi-subsets, in which elements can occur more than once. The use of these structures is very confusing and in fact unnecessary, so we omit any reference to it, including to those properties specifically referring to it.

Table 1
Sinha and Dougherty's criteria for θ, ψ, λ, ϕ to satisfy the eight axioms

(E1)	$P_{A,B} \subseteq \mathscr{H} \Leftrightarrow \theta(P_{A,B}) = 1$
(E2)	$v \in P_{A,B} \Leftrightarrow \theta(P_{A,B}) = 0$
(E3)	$(\forall \Omega_1, \Omega_2 \in \mathscr{P}([0,1]))((\forall \omega_1 \in \Omega_1)(\exists \omega_2 \in \Omega_2)(\omega_1 \leqslant \omega_2) \Rightarrow \theta(\Omega_1) \leqslant \theta(\Omega_2))$
(E4)	$(\forall x_1, x_2, y \in [0,1])(x_1 \leqslant x_2 \Rightarrow \psi(x_1, y) \leqslant \psi(x_2, y))$
(E5)	$(\forall x, y_1, y_2 \in [0,1])(y_1 \leqslant y_2 \Rightarrow \psi(x, y_1) \leqslant \psi(x, y_2))$
(E6)	$(\forall x, y \in [0,1])(x \leqslant y \Rightarrow \phi(x) \leqslant \phi(y))$
(E7)	$(\forall x, y \in [0,1])(x \leqslant y \Rightarrow \lambda(x) \geqslant \lambda(y))$
(E8)	$(\forall x, y \in [0,1])(\psi(x, y) = \psi(y, x))$
(E9)	$(\forall \Omega_1, \Omega_2 \in \mathscr{P}([0,1]))(\theta(\{\min(\omega_1, \omega_2) \mid (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2\}) = \min(\theta(\Omega_1), \theta(\Omega_2)))$
(E10)	$(\forall \alpha, \beta, \gamma \in [0,1])(\psi(\min(\alpha, \beta), \gamma) = \min(\psi(\alpha, \gamma), \psi(\beta, \gamma)))$
(E11)	$(\forall \alpha, \beta, \gamma \in [0,1])(\psi(\alpha, \min(\beta, \gamma)) = \min(\psi(\alpha, \beta), \psi(\alpha, \gamma)))$
(E12)	$(\forall \alpha \in [0,1])(\phi(\alpha) = \lambda(1 - \alpha))$
(E13)	$(\forall \alpha, \beta \in [0,1])(\lambda(\max(\alpha, \beta)) = \min(\lambda(\alpha), \lambda(\beta)))$
(E14)	$(\forall \alpha, \beta \in [0,1])(\phi(\min(\alpha, \beta)) = \min(\phi(\alpha), \phi(\beta)))$

- $\psi: [0,1] \times [0,1] \to [0,1]$, e.g. maximum, minimum, bounded sum
- $\lambda: [0,1] \to [0,1]$, e.g. Zadeh complement, identity operator
- $\phi: [0,1] \to [0,1]$, e.g. Zadeh complement, identity operator.

Definition 2.2 (General S–D indicator). For fuzzy sets A and B in a universe X the S–D indicator *Inc* for fuzzy set inclusion is defined as

$$Inc(A, B) = \theta(\{\psi(\lambda(A(x)), \phi(B(x))) \mid x \in X\}).$$

To simplify our discussion the following quantities are introduced:

$$v = \psi(\lambda(1), \phi(0)),$$

$$\mathscr{H} = \{\psi(\lambda(\alpha), \phi(\beta)) \mid (\alpha, \beta) \in [0,1]^2 \text{ and } \alpha \leqslant \beta\},$$

$$\mathscr{H}' = \{\psi(\lambda(\alpha), \phi(\beta)) \mid (\alpha, \beta) \in [0,1]^2 \text{ and } \alpha > \beta\},$$

$$P_{A,B} = \{\psi(\lambda(A(x)), \phi(B(x))) \mid x \in X\}.$$

The axioms will of course restrict possible mappings θ, ψ, λ and ϕ. In Table 1, we list the criteria, i.e. the properties that Sinha and Dougherty claim to be necessary and sufficient for the functional parameters to satisfy the axioms. [2]

Some of the properties are clearly direct consequences of other ones (e.g. (E13) follows immediately from (E7)), but we will not be concerned with redundancy at this point. We conclude this

[2] The original exposition involved additional (redundant) conditions (E15)–(E17). Since they only appeared there in the treatment of the superfluous ninth axiom, we did not include them in this table.

Table 2
Necessary and sufficient conditions for each axiom according to Sinha and Dougherty

Axiom	Necessary and sufficient conditions proposed in [11]
1	(E1)
2	(E2)
3	(E3), (E4), (E5) and (E6)
4	(E3), (E4), (E5) and (E7)
5	always fulfilled
6	(E8) and (E12)
7	(E9), (E10) and (E13)
8	(E9), (E11) and (E14)

section by summarizing in Table 2, for each axiom, which properties were believed necessary and sufficient according to Ref. [11].

3. Revision of necessary and sufficient conditions for the Sinha–Dougherty axioms

In this section, we review the properties for the Sinha–Dougherty indicator w.r.t. the axioms. We start with axiom 1.

As listed in Table 2, Sinha and Dougherty claim that (E1) is sufficient for this axiom. We provide a simple counterexample to this claim.

Indeed, consider ψ, λ and ϕ defined by $\psi(x,y)=\max(x,y)$, $\lambda(x)=1-x$ and $\phi(x)=x$ for every x and y in $[0,1]$. One easily finds:

$$\mathscr{H} = \{\max(1-\alpha,\beta)\,|\,(\alpha,\beta)\in[0,1]^2 \text{ and } \alpha\leqslant\beta\} = [0,1].$$

In other words $P_{A,B}\subseteq\mathscr{H}$ holds trivially. Choosing θ as

$$\theta : \mathscr{P}([0,1]) \to [0,1]$$
$$A \mapsto 1, \quad \forall A\in\mathscr{P}([0,1]),$$

$\theta(P_{A,B})=1$ is also satisfied for any A and B. Considering the universe $X=\{x_0,x_1,x_2\}$ and fuzzy sets $A=\{(x_0,0),(x_1,0.5),(x_2,1)\}$ and $B=\{(x_0,1),(x_1,0.5),(x_2,0)\}$, we easily see that although $A\nsubseteq B$, $\theta(P_{A,B})=1$, so axiom 1 is violated.

For a set $P_{A,B}$ of ψ-values included in $\mathscr{H}$, we cannot conclude that there exist fuzzy sets A and B so that $A\subseteq B$. Only when additionally $\mathscr{H}\cap\mathscr{H}'=\emptyset$ holds, this claim is justified. We call the new property (E1′).

$$\text{(E1}'\text{)} \quad \mathscr{H}\cap\mathscr{H}'=\emptyset.$$

It turns out that (E1) and (E1′) combined give a necessary and sufficient condition for the functional parameters to have axiom 1, as the following theorem proves.

Theorem 3.1. (E1) *and* (E1′) *together are necessary and sufficient for axiom* 1.

Proof.

- *Sufficient part.*

$$\theta(P_{A,B}) = 1 \quad \Leftrightarrow \quad P_{A,B} \subseteq \mathscr{H} \quad \Leftrightarrow \quad A \subseteq B,$$

where (E1) justifies the first and (E1′) the second equivalence.
- *Necessary part.*

Suppose axiom 1 holds. Assume for the present that $\mathscr{H} \cap \mathscr{H}' \neq \emptyset$. In other words,

$$(\exists \alpha, \beta, \gamma, \delta \in [0,1])(\alpha \leqslant \beta \text{ and } \gamma > \delta \text{ and } \psi(\lambda(\alpha), \phi(\beta)) = \psi(\lambda(\gamma), \phi(\delta))).$$

Next, construct fuzzy sets A, A', B, B' in $X = \{x\}$ in the following way:

$$A(x) = \alpha, \quad B(x) = \beta,$$
$$A'(x) = \gamma, \quad B'(x) = \delta.$$

It is noted that $A \subseteq B$, so by axiom 1 and Definition 2.2 we have $\theta(P_{A,B}) = 1$. On the other hand, $A' \nsubseteq B'$, which ensures $\theta(P_{A',B'}) \neq 1$. But by construction, $P_{A,B} = P_{A',B'}$, which leads to a contradiction. In other words, (E1′) is necessary for axiom 1. It is easily verified that in that case (E1) holds as well. □

We now concentrate on axiom 2. According to [11], (E2) is sufficient to satisfy axiom 2. Again, we come up with a counterexample, setting the functional parameters as: $\theta = \inf$, $\psi(x, y) = \min(1, x + y)$, $\lambda(x) = \phi(x) = 0$ for every x and y in $[0,1]$. By simple calculation, we get $v = \min(1, \lambda(1) + \phi(0)) = 0$. The set $P_{A,B}$ becomes $\{\min(1, \lambda(A(x)) + \phi(B(x))) \mid x \in X\} = \{0\}$, assuring that $v \in P_{A,B}$ and $\theta(P_{A,B}) = 0$ are universally true, independent of A and B. So, condition (E2) holds but it is not difficult to come up with particular instances of A and B that violate axiom 2. For instance, given $X = \{x\}$, set $A(x) = 0.5$ and $B(x) = 0.7$.

The difficulties arise from the fact that from $v \in P_{A,B}$ we cannot conclude in general that $(\exists x \in X)(A(x) = 1 \text{ and } B(x) = 0)$. It is possible that the numerical value v is attained several times, not necessarily just in $(\lambda(1), \phi(0))$. The required supplementary condition (E2′) is stated as follows:

$$(\text{E2}') \quad v \in P_{A,B} \quad \Leftrightarrow \quad (\exists x \in X)(A(x) = 1 \text{ and } B(x) = 0).$$

By construction, (E2) and (E2′) together form a necessary and sufficient condition for axiom 2.

Axioms 3 and 4 are very similar in nature, therefore we will restrict our attention to the first of them. From Table 2 we learn that Sinha and Dougherty derive (E3), (E4), (E5) and (E6) as allegedly sufficient and necessary conditions for the proposed indicator to satisfy axiom 3.

While one can easily verify that axiom 3 holds for indicators satisfying those particular conditions (proving their sufficiency), the opposite is not true: there exist values (even very sensible ones!) for the functional parameters violating some of the established conditions but maintaining the axiom's validity. First, it is noted that (E5) may safely be dropped: how ψ behaves w.r.t. its first argument is irrelevant for this axiom. A more important remark comes from the observation that (E3) rules out the infimum as a valid choice for θ; in other words, the entire class of Bandler–Kohout inclusion grades as defined in the introduction, which can be easily seen to verify axiom 3, is disposed of in this way! It seems like we need some condition complementary to (E3) that accounts for mappings like the infimum. A good candidate is the following requirement (E3′):

$$(\text{E3}') \quad (\forall \Omega_1, \Omega_2 \in \mathscr{P}([0,1]))((\forall \omega_1 \in \Omega_1)(\exists \omega_2 \in \Omega_2)(\omega_2 \leqslant \omega_1) \Rightarrow \theta(\Omega_2) \leqslant \theta(\Omega_1)).$$

Theorem 3.2. (E3), (E5), (E6) *as well as* (E3′), (E5), (E6) *constitute sufficient conditions for axiom* 3.

Proof. Suppose that $B \subseteq C$ and assume that (E3), (E5) and (E6) all hold. One easily derives:

$$\begin{aligned} Inc(A,B) &= \theta(\{\psi(\lambda(A(x)), \phi(B(x))) \mid x \in X\}) \\ &\leqslant \theta(\{\psi(\lambda(A(x)), \phi(C(x))) \mid x \in X\}) \\ &= Inc(A,C). \end{aligned}$$

The same conclusion is arrived at when one presupposes (E3′), (E5), (E6). □

The above conditions are not necessary, due to the weakness of axiom 3. Indeed, we may observe that for the constant mapping $\theta(A) = 1$, $\forall A \in \mathscr{P}(X)$, axiom 3 is always fulfilled irrespective of ψ and ϕ!

The treatment of axiom 4 is very similar to the previous one. We state the results without going into the details: (E3), (E4), (E7) on one hand and (E3′), (E4), (E7) on the other hand are sufficient but not necessary for axiom 4.

As far as axiom 5 is concerned, we are lucky: with the present general form of the indicator, it holds universally. [11]

Taking axiom 6 into consideration, we can easily see that the original conditions (E8) and (E12) imposed by [11] are sufficient but not necessary. Again this results from the weakness of the axiom. Situations where either both ϕ and λ, or ψ, or θ are constant mappings all provide sources of counterexamples.

The study of axioms 7 and 8 introduces nothing new: the conditions in Table 2 are sufficient but not necessary.

Summarizing, we have refined the original S–D conditions so as to guarantee sufficiency. Trying to find necessary conditions for each axiom in isolation does not seem to be very meaningful, because of the individual weakness of some of them. On the other hand, assuming that axioms 1–8 all hold simultaneously, the present definition of the indicator still does not allow for easy-to-list necessary conditions: its formulation is ambiguous because the same mapping is arrived at for different parameter combinations. For instance, the mapping *Inc*, which happens to satisfy all S–D axioms, defined by

$$Inc(A,B) = \inf_{x \in X} \min(1, 1 - A(x) + B(x))$$

can be decomposed in two different ways:

$\theta = \inf$ $\psi(x, y) = \min(1, x + y)$ $\phi(x) = x$ $\lambda(x) = 1 - x$	$\theta = \inf$ $\psi(x, y) = \min(1, 1 - x + y)$ $\phi(x) = x$ $\lambda(x) = x$

The second decomposition shows a non-commutative ψ ((E.8) is violated) and a ϕ and λ violating condition (E.12).

From a practical point of view the ambiguity aspect might not matter so much, but theoretically speaking it complicates the derivation of necessary conditions significantly. Therefore, in the next section, we will simplify the original S–D indicator, without sacrificing its expressiveness.

4. Revision of the general Sinha–Dougherty indicator

Definition 4.1 (Revised general S–D indicator). For fuzzy sets A and B in a universe X the revised S–D indicator *Inc* for fuzzy set inclusion is defined as [3]

$$Inc(A,B) = \theta(\{\psi(A(x), B(x)) | x \in X\}).$$

To simplify our discussion the following quantities are introduced:

$$v = \psi(1, 0),$$

$$\mathscr{H} = \{\psi(\alpha, \beta) \,|\, (\alpha, \beta) \in [0,1]^2 \text{ and } \alpha \leqslant \beta\},$$

$$\mathscr{H}' = \{\psi(\alpha, \beta) \,|\, (\alpha, \beta) \in [0,1]^2 \text{ and } \alpha > \beta\},$$

$$P_{A,B} = \{\psi(A(x), B(x)) \,|\, x \in X\}.$$

It is clear that any choice of parameters for the original indicator can still be expressed with this new formula. We don't claim that it completely rules out ambiguity, but it will be much easier to check for sufficiency and necessity of conditions. In fact, taking into account this formal simplification and resolving the dependencies between the conditions in Table 2, we arrive at a much more concise set of sufficient conditions.

Theorem 4.1 (Sufficient conditions). The indicator defined by, for $A \in \mathscr{F}(X)$, $B \in \mathscr{F}(X)$:

$$Inc(A,B) = \theta(\{\psi(A(x), B(x)) \,|\, x \in X\}),$$

where θ and ψ satisfy the conditions in Table 3 *fulfills axioms* 1–8.

[3] Note that this indicator is a special case of the original S–D indicator corresponding to the choice of the identity operator for λ and ϕ.

Table 3
Sufficient conditions for θ and ψ to satisfy the 8 axioms

(C1)	$P_{A,B} \subseteq \mathscr{H} \Leftrightarrow \theta(P_{A,B}) = 1$
(C2)	$\mathscr{H} \cap \mathscr{H}' = \emptyset$
(C3)	$v \in P_{A,B} \Leftrightarrow \theta(P_{A,B}) = 0$
(C4)	$v \in P_{A,B} \Leftrightarrow (\exists x \in X)(A(x) = 1 \text{ and } B(x) = 0)$
(C5)	$(\forall \Omega_1, \Omega_2 \in \mathscr{P}([0,1]^2))((\forall \omega_1 \in \Omega_1)(\exists \omega_2 \in \Omega_2)(\omega_1 \leqslant \omega_2)$ or $(\forall \omega_2 \in \Omega_2)(\exists \omega_1 \in \Omega_1)(\omega_1 \leqslant \omega_2) \Rightarrow \theta(\Omega_1) \leqslant \theta(\Omega_2))$
(C6)	$(\forall x_1, x_2, y \in [0,1])(x_1 \leqslant x_2 \Rightarrow \psi(x_1, y) \geqslant \psi(x_2, y))$
(C7)	$(\forall x, y_1, y_2 \in [0,1])(y_1 \leqslant y_2 \Rightarrow \psi(x, y_1) \leqslant \psi(x, y_2))$
(C8)	$(\forall \Omega_1, \Omega_2 \in \mathscr{P}([0,1]))(\theta(\{\min(\omega_1, \omega_2) \mid (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2\}) = \min(\theta(\Omega_1), \theta(\Omega_2)))$
(C9)	$(\forall x, y \in [0,1])(\psi(x, y) = \psi(1 - y, 1 - x))$

Proof. Since every inclusion indicator that can be obtained through a specific choice of parameters within the traditional S–D framework, is also covered by our revised definition, and vice versa, the derivation of sufficient conditions consists merely of tailoring the ones established in the previous section to the specific form of Definition 4.1, and ruling out the redundant ones. We also note that for brevity we have joined the separate conditions (E3) and (E3′) into the single equivalent requirement (C5).

To illustrate how this transformation process works, we focus on axiom 6, for example. From the last section we know that in the traditional framework a commutative ψ and any ϕ and λ linked by the duality condition $\phi(1-x) = \lambda(x)$, for every $x \in [0,1]$ suffice to have axiom 6. It is clear that condition (C9) accounts for this in the new framework. □

Theorem 4.1 conditions are still not necessary. Indeed, consider θ and ψ defined as

$$\theta : \mathscr{P}([0,1]) \to [0,1],$$

$$A \mapsto \begin{cases} 1 & \text{if } A = \{1\}, \\ 0 & \text{if } 0 \in A, \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

$$\psi : [0,1]^2 \to [0,1]$$

$$(x, y) \mapsto \begin{cases} 1 & \text{if } x \leqslant y, \\ 0 & \text{if } x = 1 \wedge y = 0, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

This indicator, satisfying all S–D axioms, happens to be somewhat special because it is not surjective, so we could ask whether this criterion should be included into the axioms. More gain is to be expected from an alternative line of reasoning: in the next section, we will confront the S–D axioms with the requirements for inclusion indicators established by Kitainik.

5. Link with Kitainik's work—necessary and sufficient characterization of the Sinha–Dougherty axioms

Independently of Sinha and Dougherty, Kitainik [7,8] developed an axiomatic approach to the treatment of fuzzy inclusion indicators which unfortunately received little attention so far in the fuzzy community. Very surprisingly, by postulating merely four requirements he captures almost the entire essence of Sinha and Dougherty's approach (and, as we will show, his results will enable us to drop three additionally superfluous S–D axioms!). For clarity, below we sum up the Kitainik requirements [4] and their link with the S–D axioms:

	Requirement	Formula	Equivalent to
(K1)	Contrapositivity	$Inc(A,B)=Inc(co\,B,co\,A)$	Axiom 6
(K2)	Distributivity	$Inc(A,B\cap C)=\min(Inc(A,B),Inc(A,C))$	Axiom 8
(K3)	Symmetry	$Inc(A,B)=Inc(S(A),S(B))$ with S a $\mathcal{F}(X)\to\mathcal{F}(X)$ mapping defined by, for $x\in X$, $S(A)(x)=A(s(x))$, s an $X\to X$ mapping	Axiom 5
(K4)	Heritage	Applying *Inc* to crisp sets, it coincides with crisp set inclusion	Axioms 1 and 2 for *crisp* sets

Kitainik proved that whenever a fuzzy inclusion indicator satisfies the above requirements of contrapositivity, distributivity, symmetry and heritage, Sinha and Dougherty's axioms 3, 4 and 7 automatically hold:

Theorem 5.1. *In the Sinha–Dougherty axiom list, the axioms* 3, 4 *and* 7 *are a direct consequence of the axioms* 1, 2, 5, 6 *and* 8.

The Kitainik requirements are thus equivalent with the S–D axioms with the single exception of axioms 1 and 2, which Kitainik only imposes on crisp sets. Quintessential for the derivation of a necessary and sufficient characterization of the S–D axioms is the following theorem due to Fodor and Yager [5], who were basing themselves on a result of Kitainik in [8]:

Theorem 5.2 (Fodor and Yager [5]). *A* $\mathcal{F}(X)\times\mathcal{F}(X)\to[0,1]$ *mapping Inc satisfies* (K1), (K2), (K3) *and* (K4) *if and only if there exists a contrapositive* [5] *fuzzy implicator* $\mathcal{I}$ *such that, for all A and B in* $\mathcal{F}(X)$:

$$Inc(A,B)=\inf_{x\in X}\mathcal{I}(A(x),B(x)).$$

[4] We do not call them axioms since Kitainik associated them to the definition of a fuzzy inclusion indicator.

[5] A fuzzy implicator $\mathcal{I}$ is called contrapositive if it satisfies $\mathcal{I}(x,y)=\mathcal{I}(\mathcal{N}_{\mathcal{I}}(y),\mathcal{N}_{\mathcal{I}}(x))$ for all $x,y\in[0,1]$, with $\mathcal{N}_{\mathcal{I}}$, the induced negator of $\mathcal{I}$, defined as $\mathcal{N}_{\mathcal{I}}(x)=\mathcal{I}(x,0)$ for all $x\in[0,1]$.

Effectively, this means that the form imposed on *Inc* by the Kitainik requirements also applies to the S–D axioms, since the former are a weaker version of the latter. We also see that the only admissible indicators *Inc* belong in fact to the Bandler–Kohout class. To extend Theorem 5.2 to a sufficient and necessary characterization of the S–D axioms, it therefore suffices to complement (K1)–(K4) with a necessary and sufficient condition for axioms 1 and 2. We have already established such a condition as (C1)–(C4) in Table 3.

Unfortunately, C1–C4 are a bit abstract in their formulation, and do not refer to the form introduced in Theorem 5.2. For finite universes, however, we managed to prove an alternative necessary and sufficient condition for axioms 1 and 2 in terms of a further restriction on the fuzzy implicator $\mathscr{I}$:

Theorem 5.3. *Let X be a* finite *universe and A, B fuzzy sets in X. When Inc is defined as*

$$Inc(A,B) = \inf_{x\in X} \mathscr{I}(A(x),B(x))$$

with $\mathscr{I}$ a contrapositive fuzzy implicator, then Inc satisfies axioms 1 *and* 2 *if and only if $\mathscr{I}$ also satisfies*

$$(I1) \quad (\forall x,y\in[0,1])(x\leqslant y \Leftrightarrow \mathscr{I}(x,y)=1),$$

$$(I2) \quad (\forall x,y\in[0,1])(x=1\wedge y=0 \Leftrightarrow \mathscr{I}(x,y)=0).$$

Proof. We find successively:

$$\begin{aligned} A\subseteq B &\Leftrightarrow (\forall x\in X)(A(x)\leqslant B(x)) \\ &\Leftrightarrow (\forall x\in X)(\mathscr{I}(A(x),B(x))=1) \\ &\Leftrightarrow \inf_{x\in X}\mathscr{I}(A(x),B(x))=1 \end{aligned}$$

and

$$\begin{aligned} (\exists x\in X)(A(x)=1\wedge B(x)=0) &\Leftrightarrow (\exists x\in X)(\mathscr{I}(A(x),B(x))=0) \\ &\Leftrightarrow \inf_{x\in X}\mathscr{I}(A(x),B(x))=0. \end{aligned}$$

In the last step, we have to assume the universe is finite to retain the equivalence. The above derivations show the conditions' sufficiency. It is easily verified that they are also necessary for the axioms. □

Theorems 5.2 and 5.3 taken together naturally result in the following necessary and sufficient characterization of the S–D axioms for finite universes:

Theorem 5.4. *Let X be a* finite *universe. A $\mathscr{F}(X)\times\mathscr{F}(X)\to[0,1]$ mapping Inc satisfies all Sinha–Dougherty axioms if and only if there exists a contrapositive fuzzy implicator $\mathscr{I}$ satisfying properties $(I1)$ and $(I2)$, such that, for all A and B in $\mathscr{F}(X)$:*

$$Inc(A,B) = \inf_{x\in X} \mathscr{I}(A(x),B(x)).$$

The supplementary conditions (I1) and (I2) on $\mathscr{I}$ rule out a lot of candidates for the fuzzy implicator. One suitable mapping is the Łukasiewicz implicator $\mathscr{I}_a$, defined by, for x and y in $[0,1]$:

$$\mathscr{I}_a(x,y) = \min(1, 1-x+y).$$

The above mapping can be generalized to a class of appropriate fuzzy implicators that we shall call generalized Łukasiewicz implicators.

Definition 5.1 (Generalized Łukasiewicz implicator). Every implicator $\mathscr{I}$ defined as, for x and y in $[0,1]$,

$$\mathscr{I}(x,y) = \min(1, \lambda(x) + \lambda(1-y)),$$

where λ is a strictly decreasing $[0,1] \to [0,1]$ mapping satisfying $\lambda(0)=1$, $\lambda(1)=0$ and

$$(\forall x, y \in [0,1])(x \leqslant y \Leftrightarrow \lambda(x) + \lambda(1-y) \geqslant 1) \tag{1}$$

is called a generalized Łukasiewicz implicator.

Note that the applicable restrictions are all accounted for by this definition. Every member of the class is contrapositive; (I1) is equivalent to condition (1); (I2) is fulfilled because λ strictly decreases.

The above class defined in Definition 5.1 is in agreement with the result obtained by Burillo et al. in [2]. They proved that the more specific formula

$$\inf_{x \in X} \min(1, \lambda(A(x)) + \lambda(1 - B(x)))$$

introduced by Sinha and Dougherty, satisfies all S–D axioms if and only if λ is a strictly decreasing mapping for which $\lambda(0)=1$, $\lambda(1)=0$ and condition (1) hold, i.e. they obtain precisely those indicators based on generalized Łukasiewicz implicators. An example of an indicator satisfying all Sinha–Dougherty axioms that is not based on generalized Łukasiewicz implicators is given by the following:

$$Inc(A,B) = \inf_{x \in X} \mathscr{I}(A(x), B(x))$$

with $\mathscr{I}$ given as, for x and y in $[0,1]$:

$$\mathscr{I}(x,y) = \begin{cases} 1 & \text{if } x \leqslant y, \\ 0 & \text{if } x = 1 \wedge y = 0, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

One important question which should be asked is the following: do the Kitainik requirements and the S–D axioms denote the same class of fuzzy inclusion indicators, i.e. are they equivalent? The answer is no. It suffices to come up with a fuzzy implicator $\mathscr{I}$ that is contrapositive, but does not satisfy (I1) or (I2). Such a fuzzy implicator is for example the Kleene–Dienes implicator $\mathscr{I}_{\mathrm{KD}}$, defined by, for x and y in $[0,1]$

$$\mathscr{I}_{\mathrm{KD}}(x,y) = \max(1-x, y)$$

which harms (I1). Thus, the class of inclusion indicators satisfying all S–D axioms is a strict subclass of the indicators satisfying the Kitainik requirements (assuming that the universe is finite).

6. Conclusion

We critically reviewed the inclusion indicator as proposed by Sinha and Dougherty, modified it to suit better its theoretical analysis w.r.t. a given collection of axioms, and compared it with Kitainik's results to end up with a sufficient and necessary characterization of the Sinha–Dougherty axioms. We concluded that fuzzy inclusion indicators satisfying all axioms are special Bandler–Kohout inclusion grades. Future work in this area would involve finding other non-trivial inclusion indicators not based on the generalized Łukasiewicz implicators, as well as suggestions for enriching the axiom scheme with supplementary requirements, so as to enable an even more fine-grained differentiation of suitable inclusion indicators.

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References

[1] W. Bandler, L. Kohout, Fuzzy power sets and fuzzy implication operators, Fuzzy Sets and Systems 4 (1980) 13–30.
[2] P. Burillo, N. Frago, R. Fuentes, Inclusion grade and fuzzy implication operators, Fuzzy Sets and Systems 114 (2000) 417–429.
[3] C. Cornelis, E. Kerre, Inclusion-based approximate reasoning, in: V. Alexandrov, J.J. Dongarra, B.A. Juliano, R.S. Renner, C.J.K. Tan (Eds.), Lecture Notes in Computer Science, vol. 2074, Springer, Berlin, 2001, pp. 200–210.
[4] C. Cornelis, E. Kerre, A fuzzy inference methodology based on the fuzzification of set inclusion, to appear in "Innovations in Intelligent Systems" (eds. A. Abraham, B. Nath) Springer Verlag (2002).
[5] J. Fodor, R. Yager, Fuzzy set theoretic operators and quantifiers, in: D. Dubois, H. Prade (Eds.), Fundamentals of Fuzzy Sets, Kluwer, Boston, MA, 2000, pp. 125–193.
[6] N. Frago, Morfologia matematica borrosa basada en operadores generalizados de Łukasiewicz: procesiamento de imagines, Ph.D. Thesis, Universidad publica de Navarra, 1996.
[7] L. Kitainik, Fuzzy inclusions and fuzzy dichotomous decision procedures, in: J. Kacprzyk, S. Orlovski (Eds.), Optimization Models Using Fuzzy Sets and Possibility Theory, Reidel, Dordrecht, 1987, pp. 154–170.
[8] L. Kitainik, Fuzzy Decision Procedures With Binary Relations, Kluwer, Dordrecht, 1993.
[9] B. Kosko, Neural Networks and Fuzzy Systems: A Dynamical Systems Approach to Machine Intelligence, Prentice-Hall, Englewood Cliffs, NJ, 1992.
[10] D. Ruan, E.E. Kerre, Fuzzy implication operators and generalized fuzzy method of cases, Fuzzy Sets and Systems 54 (1993) 23–37.
[11] D. Sinha, E.R. Dougherty, Fuzzification of set inclusion: theory and applications Fuzzy Sets and Systems 55 (1993) 15–42.
[12] V.R. Young, Fuzzy subsethood, Fuzzy Sets and Systems 77 (1996) 371–384.
[13] L.A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965) 338–353.