Square and Triangle: Reflections on Two Prominent Mathematical Structures for the Representation of Imprecision

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Abstract

In this paper, we study, from a predominantly syntactical viewpoint, some of the characteristics of and differences between the evaluation structures of intuitionistic fuzzy set theory ("triangle") and fuzzy four–valued or Belnap logic ("square").

Keywords: intuitionistic fuzzy sets, fuzzy four–valued logic, intuitionistic fuzzy interpretation triangle, L–fuzzy sets

1 Introduction

IFS theory basically enriches Zadeh's fuzzy set theory with a notion of indeterminacy expressing hesitation or abstention. While in the latter, membership degrees, identifying the degree to which an object satisfies a given property (generally speaking), are taken to be exact, in the former extra information in the guise of a non–membership degree is permitted to address a commonplace feature of uncertainty. In other words, IFS theory defies the claim that from the fact that an element $x \in X$ "belongs" to a given degree (say $\mu_A(x)$) to a fuzzy set A, naturally follows that x should "not belong" to A to the extent $1 - \mu_A(x)$. On the contrary, IFSs assign to each element of the universe both a degree of membership $\mu_A(x)$ and one of non–membership $\nu_A(x)$ such that $\mu_A(x) + \nu_A(x) \leq 1$, thus relaxing the enforced duality $\nu_A(x) = 1 - \mu_A(x)$ from fuzzy set theory. The amount of indeterminacy, or "missing information", is quantified by the degree $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ for all $x \in X$.

Just like the relationship between classical logic and set theory was exploited in fuzzy set theory to define "fuzzy logics" (in a narrow sense), so we may also introduce a notion of "intuitionistic fuzzy (IF) logics"; with a proposition P a degree of truth μ_P and one of falsity ν_P may be associated, such that $\mu_P + \nu_P \leq 1$. This idea is elaborated in e.g. [1].

One way to generalize IFS theory is to drop the restriction that $\mu_A(x) + \nu_A(x) \leq 1$, and instead draw $(\mu_A(x), \nu_A(x))$, or (μ_P, ν_P), from $[0, 1]^2$. This extension was coined fuzzy four–valued logic, and is sometimes also referred to as fuzzy Belnap logic, in reference to the logical evaluation structure $\mathcal{FOUR}$ introduced by Belnap [3] and shown in Figure 1.

Fuzzy four–valued logic extends $\mathcal{FOUR}$ by drawing values from the entire unit square and not just from its angular points. Those angular points, incidentally, codify the epistemic

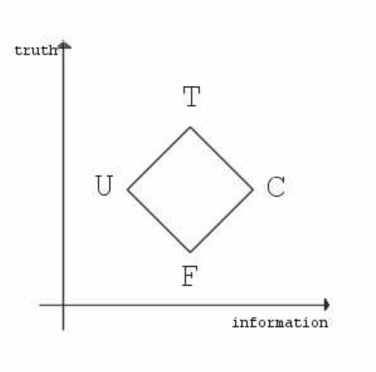

Figure 1: Belnap's logical evaluation structure $\mathcal{FOUR}$

states **true** (T), **false** (F), **unknown** (U) and **contradiction** (C) that can represent an agent's beliefs with respect to a proposition. By defining the correspondences $T \to (1,0)$, $F \to (0,1)$, $U \to (0,0)$ and $C \to (1,1)$, it is easy to perceive how this structure relates to IFS theory; in the latter, by the restriction on membership degrees/non–membership degrees (truth/falsity degrees) the state C is not allowed. As a consequence, its evaluation structure will be a triangle that takes up only (the consistent) half of the unit square.

In this paper, we compare the evaluation structures of IFS theory and fuzzy–four valued logic. The exposition will be from an L–fuzzy set theoretical perspective, i.e. the respective evaluation structures "Triangle" and "Square" are viewed as particular complete lattices. In this way, the definition of graded versions of logical connectives becomes transparant. We consider representational issues w.r.t. these connectives, and we also show that the bijections Atanassov defined between "Triangle" and "Square" are in fact not lattice isomorphisms and therefore limit the extent of useful consequences to be drawn from this perceived "equivalence" between IFS theory and fuzzy four–valued logic.

2 Evaluation Structures: the L–Fuzzy Set Perspective

The defining idea behind our approach is to treat logical connectives as algebraic mappings; to describe the domain and codomain structure for intuitionistic fuzzy connectives the partially ordered set $(L^*, \leq_{L^*})$ was introduced in [4]:

Definition 1 **($(L^*, \leq_{L^*})$, "Triangle")**

$$L^* = \{(x_1, x_2) \in [0,1]^2 \mid x_1 + x_2 \leq 1\}$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \wedge x_2 \geq y_2$$

It is easily verified that $(L^*, \leq_{L^*})$ is a complete lattice. By $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$ we denote its bounds. A graphical representation of L^* is the intuitionistic fuzzy representation triangle (shortly, "Triangle") shown in Figure 2. An IFS in X may simply be defined as an L^*–fuzzy set in X, i.e. a mapping from X to L^* such that $A(x) = (\mu_A(x), \nu_A(x))$ for each $x \in X$.

For fuzzy four–valued logic, the following lattice $(L_\Box, \leq_\Box)$ can be introduced:

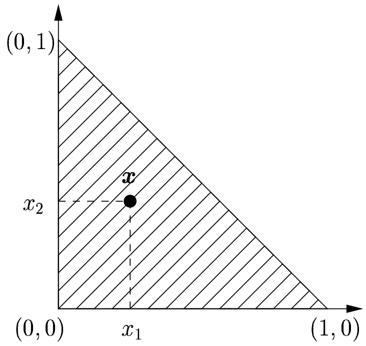

Figure 2: Intuitionistic Fuzzy Interpretation Triangle

Definition 2 ($(L_\Box, \leq_\Box)$, "Square")

$$L_\Box = [0, 1]^2$$

$$(x_1, x_2) \leq_\Box (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \wedge x_2 \geq y_2$$

By $0_\Box = (0, 1)$ and $1_\Box = (1, 0)$ we denote the bounds of the complete lattice $((L_\Box, \leq_\Box)$. Its graphical interpretation is shown in Figure 3.

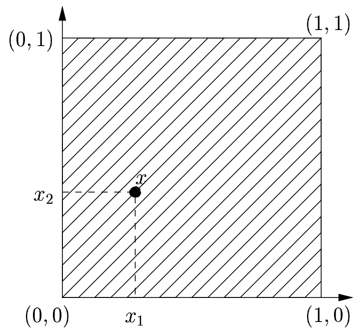

Figure 3: Graphical representation of $(L_\Box, \leq_\Box)$

3 Graded Logical Connectives

In the next three subsections, we recall and establish some important results w.r.t. the definition of extensions of the negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$) and implication ($\rightarrow$) connectives from classical logic.

3.1 Negation

Atanassov [1] defined the negation of an element $(x_1, x_2) \in L^*$ as (x_2, x_1). In [8, 9] a more general definition encapsulating the former was given:

Definition 3 (Negator on L^*) *A negator on L^* is any decreasing $L^* \rightarrow L^*$–mapping $\mathcal{N}$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}, \mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x, \forall x \in L^*$, then $\mathcal{N}$ is called an involutive negator.*

The mapping $\mathcal{N}_s$, defined as $\mathcal{N}_s(x_1, x_2) = (x_2, x_1)$ will be called the *standard negator*. The following theorem was established in [8]:

Theorem 1 *Let $\mathcal{N}$ be a negator on L^*, and let the $[0,1] \to [0,1]$–mapping N be defined by, for $a \in [0,1]$, $N(a) = pr_1\mathcal{N}(a, 1-a)$, with $pr_1(x)$ denoting the first component of $x \in L^*$. Then $\mathcal{N}$ is involutive if and only if N is involutive and for all $(x_1, x_2) \in L^*$:*

$$\mathcal{N}(x_1, x_2) = (N(1-x_2), 1 - N(x_1)).$$

Definition 4 (Negator on $L_\Box$) *A negator on $L_\Box$ is any decreasing $L_\Box \to L_\Box$–mapping $\mathfrak{N}$ satisfying $\mathfrak{N}(0_\Box) = 1_\Box$ and $\mathfrak{N}(1_\Box) = 0_\Box$. If $\mathfrak{N}(\mathfrak{N}(x)) = x, \forall x \in L_\Box$, then $\mathfrak{N}$ is called an involutive negator.*

Lemma 1 *For any involutive negator $\mathfrak{N}$ on $L_\Box$ one of the following holds:*

(i) $\mathfrak{N}(0,0) = (0,0)$ *and* $\mathfrak{N}(1,1) = (1,1)$*; or*

(ii) $\mathfrak{N}(0,0) = (1,1)$ *and* $\mathfrak{N}(1,1) = (0,0)$.

Theorem 2 *Let $\mathfrak{N}$ be a negator on $L_\Box$.*

(i) If $\mathfrak{N}(0,0) = (0,0)$, then let φ be the $[0,1] \to [0,1]$–mapping defined as $\varphi(a) = pr_1\mathfrak{N}(0,a)$, for all $a \in [0,1]$. Then $\mathfrak{N}$ is involutive if and only if φ is an increasing permutation of $[0,1]$ and, for all $(x_1, x_2) \in L_\Box$,

$$\mathfrak{N}(x_1, x_2) = (\varphi(x_2), \varphi^{-1}(x_1)).$$

(ii) If $\mathfrak{N}(0,0) = (1,1)$, then let N_1 and N_2 be the $[0,1] \to [0,1]$–mappings defined as $N_1(a) = pr_1\mathfrak{N}(a,0)$ and $N_2(a) = pr_2\mathfrak{N}(0,a)$, for all $a \in [0,1]$. Then $\mathfrak{N}$ is involutive if and only if N_1 and N_2 are involutive negators on $[0,1]$ and, for all $(x_1, x_2) \in L_\Box$,

$$\mathfrak{N}(x_1, x_2) = (N_1(x_1), N_2(x_2)).$$

If in the first case $\varphi(x_1) = x_1$, for all $x_1 \in [0,1]$, then we obtain $\mathfrak{N}(x_1, x_2) = (x_2, x_1)$, i.e. the straightforward extension of the standard negation $\mathcal{N}_s$ on L^* to $L_\Box$. We denote this negator by $\mathfrak{N}_s^1$. If in the second case $N_1(x_1) = N_2(x_1) = 1 - x_1$, for all $x_1 \in [0,1]$, then we obtain $\mathfrak{N}(x_1, x_2) = (1 - x_1, 1 - x_2)$. We denote this negator by $\mathfrak{N}_s^2$.

Let $N(a) = \varphi(1-a)$, for all $a \in [0,1]$, then N is a bijective negator on $[0,1]$ and in Theorem 2(i) we obtain $\mathfrak{N}(x_1, x_2) = (N(1-x_2), 1 - N^{-1}(x_1))$, for all $(x_1, x_2) \in L_\Box$. While for negators on L^* the corresponding negator N is involutive, this is not necessarily the case for negators on $L_\Box$. Note also that the case (ii) in Theorem 2 cannot occur in L^*.

4 Conjunction and Disjunction

Since $\leq_{L^*}$ is a partial ordering, an order–theoretical definition of conjunction and disjunction on L^* as triangular norms and conorms, t–norms and t–conorms for short, respectively, arises quite naturally:

Definition 5 (Triangular Norm on L^*) *A t–norm on L^* is any increasing, commutative, associative $(L^*)^2 \to L^*$–mapping $\mathcal{T}$ satisfying $\mathcal{T}(1_{L^*}, x) = x$, for all $x \in L^*$.*

Definition 6 (Triangular Conorm on L^*) *A t–conorm on L^* is any increasing, commutative, associative $(L^*)^2 \to L^*$–mapping $\mathcal{S}$ satisfying $\mathcal{S}(0_{L^*}, x) = x$, for all $x \in L^*$.*

Involutive negators on L^* are always linked to an associated fuzzy connective (a negator on $[0,1]$); the same does not always hold true for t–norms and t–conorms, however. We therefore have to introduce the following definition: [5]

Definition 7 (t–representability) *A t–norm $\mathcal{T}$ on L^* (resp. t–conorm $\mathcal{S}$) is called t–representable if there exists a t–norm T and a t–conorm S on $[0,1]$ (resp. a t–conorm S' and a t–norm T' on $[0,1]$) such that, for $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,*

$$\begin{aligned}\mathcal{T}(x,y) &= (T(x_1,y_1), S(x_2,y_2)),\\ \mathcal{S}(x,y) &= (S'(x_1,y_1), T'(x_2,y_2)).\end{aligned}$$

T and S (resp. S' and T') are called the representants of $\mathcal{T}$ (resp. $\mathcal{S}$).

The theorem below states the conditions under which a pair of connectives on $[0,1]$ gives rise to a t–representable t–norm (t–conorm) on L^*.

Theorem 3 *[5] Given a t–norm T and t–conorm S on $[0,1]$ satisfying $T(a,b) \le 1 - S(1-a, 1-b)$ for all $a, b \in [0,1]$, the mappings $\mathcal{T}$ and $\mathcal{S}$ defined by, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in L^*:*

$$\begin{aligned}\mathcal{T}(x,y) &= (T(x_1,y_1), S(x_2,y_2)),\\ \mathcal{S}(x,y) &= (S(x_1,y_1), T(x_2,y_2)),\end{aligned}$$

are a t–norm and a t–conorm on L^*, respectively.

The dual of a t–norm $\mathcal{T}$ on L^* (t–conorm $\mathcal{S}$) w.r.t. a negator $\mathcal{N}$ is the mapping $\mathcal{T}^*$ (resp. $\mathcal{S}^*$) defined by, for $x, y \in L^*$,

$$\mathcal{T}^*(x,y) = \mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y))) \text{ (resp. } \mathcal{S}^*(x,y) = \mathcal{N}(\mathcal{S}(\mathcal{N}(x), \mathcal{N}(y)))).$$

It can be verified that $\mathcal{T}^*$ is a t–conorm and $\mathcal{S}^*$ is a t–norm on L^*. Moreover, the dual t–norm (t–conorm) with respect to an involutive negator $\mathcal{N}$ on L^* of a t–representable t–conorm (t–norm) is t–representable. [9]

In [9] a representation theorem for t–norms on L^* meeting a number of criteria was formulated and proven.

Theorem 4 *$\mathcal{T}$ is a continuous t–norm on L^* satisfying*

- $(\forall x \in L^* \setminus \{0_{L^*}, 1_{L^*}\})(\mathcal{T}(x,x) <_{L^*} x)$ *(archimedean property)*
- $(\exists x, y \in L^*)(x_1 \neq 0$ *and* $x_2 \neq 0$ *and* $y_1 \neq 0$ *and* $y_2 \neq 0$ *and* $\mathcal{T}(x,y) = 0_{L^*})$ *(strong nilpotency)*
- $(\forall x, y, z \in L^*)(\mathcal{T}(x,z) \le_{L^*} y \Leftrightarrow z \le_{L^*} \sup\{\gamma \in L^* \mid \mathcal{T}(x,\gamma) \le_{L^*} y\})$ *(residuation principle)*
- $(\forall x, y \in D)(\sup\{\gamma \in L^* \mid \mathcal{T}(x,\gamma) \le_{L^*} y\} \in D)$
- $\mathcal{T}((0,0),(0,0)) = 0_{L^*}$

if and only if there exists an increasing continuous permutation φ of $[0,1]$ such that, for all $x, y \in L^*$,

$$\mathcal{T}(x,y) = (\varphi^{-1}(\max(0, \varphi(x_1) + \varphi(y_1) - 1)), 1 - \varphi^{-1}(\max(0, \varphi(x_1) + \varphi(1-y_2) - 1, \varphi(y_1) + \varphi(1-x_2) - 1))),$$

or equivalently, there exists a continuous increasing permutation Φ of L^* with increasing inverse such that $\mathcal{T} = \Phi^{-1} \circ \mathcal{T}_W \circ (\Phi \circ pr_1, \Phi \circ pr_2)$, where $\mathcal{T}_W$, the Łukasiewicz t–norm on L^*, is defined by, for $x, y \in L^*$:

$$\mathcal{T}_W(x,y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + 1 - y_1, y_2 + 1 - x_1)).$$

Definition 8 (Triangular Norm on $L_\square$) *A t–norm on $L_\square$ is any increasing, commutative, associative $(L_\square)^2 \to L_\square$–mapping $\mathfrak{T}$ satisfying $\mathfrak{T}(1_\square, x) = x$, for all $x \in L_\square$.*

Definition 9 (Triangular Conorm on $L_\square$) *A t–conorm on $L_\square$ is any increasing, commutative, associative $(L_\square)^2 \to L_\square$–mapping $\mathfrak{S}$ satisfying $\mathfrak{S}(0_\square, x) = x$, for all $x \in L_\square$.*

t–representability is defined in a similar way as for t–norms and t–conorms on L^*. In [11] examples of t–norms on $L_\square$ are given which are not t–representable. Also the dual t–conorm of a t–norm on $L_\square$ w.r.t. to a negator on $L_\square$ is defined in a similar way as for t–norms on L^*, and similarly for the dual t–norm.

The following are examples of t–representable t–norms and t–conorms on $L_\square$, for $x, y \in L_\square$:

- $\inf(x,y) = (\min(x_1, y_1), \max(x_2, y_2))$,
- $\mathfrak{T}_W(x,y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + y_2))$,
- $\sup(x,y) = (\max(x_1, y_1), \min(x_2, y_2))$,
- $\mathfrak{S}_W(x,y) = (\min(1, x_1 + y_1), \max(0, x_2 + y_2 - 1))$.

Note that the dual of $\mathfrak{T}_W$ w.r.t. both $\mathfrak{N}_s^1$ and $\mathfrak{N}_s^2$ is equal to $\mathfrak{S}_W$, i.e.

$$\mathfrak{S}_W(x,y) = \mathfrak{N}_s^1(\mathfrak{T}_W(\mathfrak{N}_s^1(x), \mathfrak{N}_s^1(y))) = \mathfrak{N}_s^2(\mathfrak{T}_W(\mathfrak{N}_s^2(x), \mathfrak{N}_s^2(y))).$$

In [9] we introduced the residuation principle for t–norms on L^* as follows: a t–norm satisfies the residuation principle if and only if, for all $x, y, z \in L^*$,

$$\mathcal{T}(x,y) \leq_{L^*} z \Leftrightarrow y \leq_{L^*} \mathcal{I}_\mathcal{T}(x,z).$$

The residuation principle for t–norms on $L_\square$ can be introduced in a similar way.

In [9] we have shown that $\mathcal{T}_W$ satisfies the residuation principle. This is not the case anymore if we straightforwardly extend $\mathcal{T}_W$ to $L_\square$, $\mathcal{T}_W$ is even not a t–norm on $L_\square$. Moreover, De Baets and Mesiar proved in [7] that if a t–norm T on a complete product lattice $L = L_1 \times L_2$ satisfies the residuation principle, then T is the direct product of two t–norms on L_1 and L_2, respectively. This result can be translated in our terminology as follows.

Theorem 5 *Any t–norm $\mathfrak{T}$ on $L_\square$ satisfying the residuation principle is t–representable.*

Note that this result does not hold in L^*: $\mathcal{T}_W$ satisfies the residuation principle but is not t–representable!

5 Implication

A very general definition of the implication connective on L^* is given in the following definition [4]:

Definition 10 (Implicator on L^*) *An implicator on L^* is any $(L^*)^2 \rightarrow L^*$–mapping $\mathcal{I}$ satisfying $\mathcal{I}(0_{L^*}, 0_{L^*}) = 1_{L^*}, \mathcal{I}(1_{L^*}, 0_{L^*}) = 0_{L^*}, \mathcal{I}(0_{L^*}, 1_{L^*}) = 1_{L^*}, \mathcal{I}(1_{L^*}, 1_{L^*}) = 1_{L^*}$. Moreover we require $\mathcal{I}$ to be decreasing in its first, and increasing in its second component.*

Two important subclasses of implicators on L^* were introduced in [5]. It is easily verified that each of the mappings defined hereafter is indeed an implicator in the sense of Definition 10.

Definition 11 (S–implicator) *Let $\mathcal{S}$ be an t–conorm on L^* and $\mathcal{N}$ a negator on L^*. The S–implicator generated by $\mathcal{S}$ and $\mathcal{N}$ is the mapping $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ defined as, for $x, y \in L^*$:*

$$\mathcal{I}_{\mathcal{S},\mathcal{N}}(x, y) = \mathcal{S}(\mathcal{N}(x), y).$$

If $\mathcal{S}$ is t–representable, $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is called a t–representable S–implicator.

Definition 12 (R–implicator) *Let $\mathcal{T}$ be an t–norm on L^*. The R–implicator generated by $\mathcal{T}$ is the mapping $\mathcal{I}_{\mathcal{T}}$ defined as, for $x, y \in L^*$:*

$$\mathcal{I}_{\mathcal{T}}(x, y) = \sup\{\gamma \in L^* \mid \mathcal{T}(x, \gamma) \leq_{L^*} y\}.$$

If $\mathcal{T}$ is t–representable, $\mathcal{I}_{\mathcal{T}}$ is called a t–representable R–implicator.

The R–implicator generated by $\mathcal{T}_W$ is equal to the S–implicator generated by $\mathcal{S}_W$ and $\mathcal{N}_s$, where $\mathcal{S}_W$ denotes the dual t–conorm of $\mathcal{T}_W$ w.r.t. $\mathcal{N}_s$, i.e. for $x, y \in L^*$,

$$\mathcal{I}_{\mathcal{T}_W}(x, y) = \mathcal{I}_{\mathcal{S}_W,\mathcal{N}_s}(x, y) = (\min(1, y_1 + 1 - x_1, x_2 + 1 - y_2), \max(0, y_2 + x_1 - 1)).$$

This result does not hold for the t–representable extension $\mathcal{T}_w$ of the Łukasiewicz t–norm on $[0, 1]$ to L^*, defined as, for $x, y \in L^*$,

$$\mathcal{T}_w(x, y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + y_2).$$

Indeed, we have, for $x, y \in L^*$,

$$\begin{aligned} \mathcal{I}_{\mathcal{T}_w}(x, y) &= (\min(1, y_1 + 1 - x_1, x_2 + 1 - y_2), \max(0, y_2 - x_2)), \\ \mathcal{I}_{\mathcal{T}_w^*,\mathcal{N}_s}(x, y) &= (\min(1, x_2 + y_1), \max(0, x_1 + y_2 - 1)). \end{aligned}$$

Definition 13 (Implicator on $L_\square$) *An implicator on $L_\square$ is any $(L_\square)^2 \rightarrow L_\square$–mapping $\mathfrak{I}$ satisfying $\mathfrak{I}(0_\square, 0_\square) = \mathfrak{I}(0_\square, 1_\square) = \mathfrak{I}(1_\square, 1_\square) = 1_\square$ and $\mathfrak{I}(0_\square, 0_\square) = 0_\square$. Moreover we require $\mathfrak{I}$ to be decreasing in its first, and increasing in its second component.*

The notions of S–implicator and R–implicator on $L_\square$ are defined in a similar way as in L^*. The R–implicator generated by $\mathfrak{T}_W$ is equal to the S–implicator generated by $\mathfrak{S}_W$ and $\mathfrak{N}_s^2$, i.e. for $x, y \in L_\square$,

$$\mathfrak{I}_{\mathfrak{T}_W}(x, y) = \mathfrak{I}_{\mathfrak{S}_W,\mathfrak{N}_s^2}(x, y) = (\min(1, y_1 + 1 - x_1), \max(0, y_2 - x_2)).$$

This implicator is however not equal to the S–implicator generated by $\mathfrak{S}_W$ and $\mathfrak{N}_s^1$, which is given by, for $x, y \in L_\Box$,

$$\mathfrak{I}_{\mathfrak{S}_W,\mathfrak{N}_s^1}(x,y) = (\min(1, x_2 + y_1), \max(0, x_1 + y_2 - 1).$$

From the above it follows that the equality between the R–implicator of and the corresponding S–implicator holds in $L_\Box$ for the t–representable t–norm $\mathfrak{T}_W$, but in L^* the equality holds for the non–t–representable t–norm $\mathcal{T}_W$ and not for the t–representable $\mathcal{T}_w$.

The suitability of implicators on L^* for a variety of purposes can be assessed using the (generalized) criteria introduced by Smets and Magrez in [12]:

Definition 14 (Axioms of Smets and Magrez for an implicator $\mathcal{I}$ on L^*)

(A.1) $(\forall y \in L^*)(\mathcal{I}(.,y)$ *is decreasing in* $L^*)$
$(\forall x \in L^*)(\mathcal{I}(x,.)$ *is increasing in* $L^*)$ (monotonicity laws)

(A.2) $(\forall x \in L^*)(\mathcal{I}(1_{L^*}, x) = x)$ (neutrality principle)

(A.3) $(\forall (x,y) \in (L^*)^2)(\mathcal{I}(x,y) = \mathcal{I}(\mathcal{N}_\mathcal{I}(y), \mathcal{N}_\mathcal{I}(x))$ (contrapositivity)

(A.4) $(\forall (x,y,z) \in (L^*)^3)(\mathcal{I}(x, \mathcal{I}(y,z)) = \mathcal{I}(y, \mathcal{I}(x,z)))$ (interchangeability principle)

(A.5) $(\forall (x,y) \in (L^*)^2)(x \leq_{L^*} y \Leftrightarrow \mathcal{I}(x,y) = 1_{L^*})$ (confinement principle)

(A.6) $\mathcal{I}$ *is a continuous* $(L^*)^2 \to L^*$*–mapping* (continuity)

The axioms of Smets and Magrez for an implicator on $L_\Box$ are introduced in a similar way.

In [9] it is proven that $\mathcal{I}_{\mathcal{T}_W}$ satisfies all six Smets–Magrez axioms. Furthermore no t–representable S–implicator nor t–representable R–implicator satisfies all six axioms. On the other hand, in $L_\Box$ we have that $\mathfrak{I}_{\mathfrak{T}_W}$ satisfies all Smets–Magrez axioms. In other words, t–representability plays very different roles in the intuitionistic fuzzy and in the fuzzy four–valued framework!

6 Relationship Between the Triangle and the Square

In [2], Atanassov defined two transformations F and G from $L_\Box$ to L^*, defined for $(x_1, x_2) \in [0,1]^2$ by

$$F(x_1,x_2) = \begin{cases} (0,0) & \text{if } x_1 = x_2 = 0 \\ \left(\frac{x_1^2}{x_1+x_2}, \frac{x_1 x_2}{x_1+x_2}\right) & \text{if } x_1 \geq x_2 \\ \left(\frac{x_1 x_2}{x_1+x_2}, \frac{x_2^2}{x_1+x_2}\right) & \text{if } x_1 < x_2 \end{cases} \quad (1)$$

$$G(x_1,x_2) = \begin{cases} \left(x_1 - \frac{x_2}{2}, \frac{x_2}{2}\right) & \text{if } x_1 \geq x_2 \\ \left(\frac{x_1}{2}, x_2 - \frac{x_1}{2}\right) & \text{if } x_1 < x_2 \end{cases} \quad (2)$$

and proved that they are bijective, showing that each L–fuzzy set with a lattice L that can be represented in the form of figure 3, can be represented as an IFS, too. It is important to realize, however, that the transformations are not lattice isomorphisms, i.e. they do not satisfy

$$\begin{aligned} F(\inf(x,y)) &= \inf(F(x), F(y)) \\ F(\sup(x,y)) &= \sup(F(x), F(y)) \\ G(\inf(x,y)) &= \inf(G(x), G(y)) \\ G(\sup(x,y)) &= \sup(G(x), G(y)) \end{aligned}$$

For instance, take $x = (1,1)$, $y = (0,0)$, so $\inf(x,y) = (0,1)$ in $(L_\square, \leq_\square)$. Now $F(0,1) = (0,1)$, while $\inf(F(1,1), F(0,0)) = \inf\left(\left(\frac{1}{2},\frac{1}{2}\right),(0,0)\right) = \left(0,\frac{1}{2}\right)$.

As a consequence, the transformations do not preserve the order either, e.g. $x \leq_\square y \not\Leftrightarrow F(x) \leq_{L^*} F(y)$, so order–theoretical concepts like negators, t–norms, t–conorms and implicators are not transferred by them; this is also confirmed by our results in the previous section, which show that "Square" and "Triangle" have quite different characteristics.

7 Conclusion

This paper has hinted at some of the distinguishing features between (the evaluation structures of) intuitionistic fuzzy sets and fuzzy four–valued logic. It is especially remarkable how t–representability which acts as a key concept in IFS theory has only a marginal role to play in the setting of fuzzy four–valued logic.

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