A Semantical Approach to Rough Sets and Dominance-Based Rough Sets

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Abstract. There exist two formulations of rough sets: the conceptual and computational one. The conceptual or semantical approach of rough set theory focuses on the meaning and interpretation of concepts, while algorithms to compute those concepts are studied in the computational formulation. However, the research on the former is rather limited. In this paper, we focus on a semantically sound approach of Pawlak's rough set model and covering-based rough set models. Furthermore, we illustrate that the dominance-based rough set model can be rephrased using this semantic approach.

Keywords: Covering-based rough sets · Dominance-based rough sets · Semantics · Pre-order

1 Introduction

Rough set theory has two formulations: a conceptual and a computational one [29]. The former formulation emphasizes the meaning and interpretation of concepts and notions of the theory, while the latter formulation is used to construct procedures and algorithms to compute those notions. A major difference between the formulations is the notion of definability. Nevertheless, both formulations are complementary and they are both fundamental in the research on rough set theory. In addition, it is sometimes necessary to consider both approaches together, for example in the minimal description length principle [8,16] in which it is stated that we need to find a balance between the loss of accuracy (computational) and a more compact description of data models (conceptual) when computing decision reducts.

The research on computational formulations has dominated the rough set theory research field since the seminal paper of Pawlak [10]. For instance, there is a

J.P. Carvalho et al. (Eds.): IPMU 2016, Part II, CCIS 611, pp. 23–35, 2016.
DOI: 10.1007/978-3-319-40581-0_3

broad study on generalized rough set models, in which a binary relation or neighborhood operator is used to describe the indiscernibility relation between objects of the universe [2,3,17,21,24,38]. In addition, several covering-based rough set models are defined in literature [12,19,20,22,23,30–38]. More recently, the classification and comparison of these different rough set models have been discussed [1,14,15,27].

In such contributions on the computational formulations of rough set theory, the notion of definability of subsets of the universe of discourse is determined by the approximation operators. This can be done in different ways: a subset $X \subseteq U$ is definable if X equals its lower approximation, or if X equals its upper approximation, or if its lower and upper approximation are equal.

In comparison, the contribution to the conceptual formulation of rough set theory is limited. Prior to the introduction of rough sets, Pawlak [9] and Marek and Pawlak [6] described a definable set as the extension of a concept, of which its intension is a formula in a descriptive language based on the data. The intension of a concept is an abstract description of the properties characteristic for a concept, while the extension of a concept contains all the objects of the universe of discourse satisfying those properties [29]. Hence, a concept is jointly described by its intension and extension. However, except for a few articles by Marek and Truszczyński [7], Yao and Zhou [25] and Yao [26], this notion of definability, which is semantically superior, has scarcely been discussed.

In this paper, we refocus our attention on the conceptual or semantical approach of rough sets. Given an information or decision table which represents the data, definable subsets of the universe are used as primitive notion. A definable set is an arbitrary union of elementary sets. Such an elementary set is a basic granule which represents an indivisible block of information, obtained from the table. Hence, each elementary and definable set is meaningful. In the original rough set of Pawlak [10], the elementary sets are given by the equivalence classes U/E related to the equivalence relation E, which represents the indiscernibility relation between the objects based on the given data. However, it is not always possible to construct such a partition. For example, when the data is incomplete, or when there is a strict order on the values of certain attribute values. Therefore, we extend the semantical approach of Pawlak's model to covering-based rough set models. The elementary sets are now no longer given by a partition U/E, but by a covering $\mathbb{C}$. As each definable set is the union of elementary sets, the set of definable sets is obtained by closing the set of elementary sets under set union. This results in the Boolean algebra $\mathscr{B}(U/E)$ for the model of Pawlak and in the $\cup$-closed set $\cup^*(\mathbb{C})$ for the covering-based rough set models. As the definability of sets is established, the approximation of undefinable sets by definable sets comes naturally [26]. Therefore, approximation operators are constructed as derived notions in both Pawlak's rough set model and the framework of covering-based rough set models.

To illustrate the semantic approach of covering-based rough sets, we determine the elementary and definable sets in the framework of dominance-based rough sets, introduced by Greco et al. [2,3,18]. In this framework, the indiscernibility relation between objects is given by a dominance relation or pre-order, which is a reflexive and transitive relation. Such an indiscernibility relation is useful, when the data represented in the table are preference-ordered, i.e., based on the different conditional attributes, it is determined whether an object x is preferred over an object y.

Given an object x and a dominance relation $\succeq$, two neighborhoods of x are defined by the objects dominating x and the objects dominated by x. Moreover, each set of neighborhoods results in a covering. Given both coverings, by the semantic approach of covering-based rough sets meaningful lower approximation operators are constructed. It will be shown that these conceptual lower approximation operators are exactly the computational lower approximation operators suggested by Greco et al. [2,3,18].

Furthermore, it is discussed how certain decision rules are obtained from a decision table using dominance-based rough sets. Rule induction [4] is an important technique to extract knowledge from a decision table and can be used for the classification of new objects. For example, to determine whether a new object is preferable to the current objects.

This paper is organized as follows. In Sect. 2, we discuss the semantic approach of Pawlak's rough set model and covering-based rough set models. In Sect. 3, we recall some results on the different characterizations of the lower approximation operator based on a reflexive and transitive neighborhood operator. In Sect. 4, we illustrate that the dominance-based lower approximation operator introduced by Greco et al. is semantically meaningful. Furthermore, we discuss how decision rules are obtained in the dominance-based framework. To end, we state conclusion and future work in Sect. 5.

2 Semantics of Rough Set Models

Rough set analysis is a tool to study data given in an information table. Formally, a complete information table is a tuple $T = (U, At, \{V_a \mid a \in A\}, \{I_a \mid a \in At\})$, where U is a finite non-empty set of objects, At is a finite non-empty set of attributes and for each $a \in A$, V_a represents a non-empty set of values related to the attribute a. Furthermore, the information functions $I_a \colon U \to V_a$ map every object of U to a value in V_a, for each $a \in At$. The table T is called a complete decision table if the set of attributes At consists of the disjoint sets C and $\{d\}$, where C represents the conditional attributes of the table and d represents the decision attribute. In this paper, we use the closed world assumption, i.e., the table contains all objects under consideration [11].

Given such a table T, a basic granule represents an elementary unit of knowledge we can obtain from the table and is formally given by a subset of the universe U. Given a non-empty family of granules $G \subseteq 2^U$, the poset $(G, \subseteq)$ is called a granular structure, where $\subseteq$ is the set-theoretic inclusion relation [28]. By imposing different conditions on the set G, we derive different models (U, G) of granular structures. For example, when G is closed under set intersection, set union and set complement, the model $(U, (G, \cap, \cup, {}^c))$ is a Boolean algebra [28].

In the original rough set model of Pawlak [10,11] the basic granules of the information table are given by equivalence classes. Namely, let $A \subseteq C$ be a set of conditional attributes, then we can define an equivalence relation E_A on U as follows:

$$\forall x, y \in U\colon xE_Ay \Leftrightarrow \forall a \in A\colon I_a(x) = I_a(y).$$

The equivalence class of an object x is given by $[x]_{E_A} = \{y \in U \mid xE_Ay\}$. The partition U/E_A can be seen as a family of basic granules and each equivalence class is called an *elementary set*. Moreover, the union of a family of equivalence classes is called a *definable set*, since such a set can be constructed and interpreted from the available data in the information table. The granule structure which is used in rough set theory to represent the definable sets is given by

$$\mathscr{B}(U/E_A) = \{\bigcup F \mid F \subseteq U/E_A\}$$

and is obtained by closing the partition U/E_A under set union. As $\mathscr{B}(U/E_A)$ is closed under set intersection, set union and set complement, it is a Boolean algebra. Therefore, the granule structure $(U, (\mathscr{B}(U/E_A), \cap, \cup, ^c))$ which represents the definable sets of the information table in Pawlak's rough set model can be seen as a model of granular structures, as described in [28].

However, not every subset of U is contained in the granular structure $\mathscr{B}(U/E_A)$. Such a set, which we call *undefinable*, must be approximated by subsets in the granular structure. Naturally, the lower approximation of $X \subseteq U$ consists of definable sets which are subsets of X (approximation from below), while the upper approximation of X consists of definable sets which are supersets of X (approximation from above) [10,28].

As in Pawlak's rough set model the definable sets are represented by the Boolean algebra $\mathscr{B}(U/E_A)$, there is a unique greatest set in $\mathscr{B}(U/E_A)$ contained by X, and a unique smallest set in $\mathscr{B}(U/E_A)$ containing X. Therefore, in Pawlak's rough set model the approximations of X are given by

$$\begin{aligned}
\underline{\mathrm{apr}}_A(X) &= \text{the greatest definable set in } \mathscr{B}(U/E_A) \text{ contained by } X \\
&= \bigcup \{Y \in \mathscr{B}(U/E_A) \mid Y \subseteq X\}, \\
\overline{\mathrm{apr}}_A(X) &= \text{the smallest definable set in } \mathscr{B}(U/E_A) \text{ containing } X \\
&= \bigcap \{Y \in \mathscr{B}(U/E_A) \mid X \subseteq Y\}.
\end{aligned}$$

Note that for all $X \subseteq U$, its lower and upper approximation are definable, i.e., $\underline{\mathrm{apr}}_A(X) \in \mathscr{B}(U/E_A)$ and $\overline{\mathrm{apr}}_A(X) \in \mathscr{B}(U/E_A)$. Moreover, if X is definable, then

$$\underline{\mathrm{apr}}_A(X) = \overline{\mathrm{apr}}_A(X) = X.$$

Hence, one derives the notion of definability which is used in computational formulations.

Unfortunately, it is not always possible to construct a meaningful equivalence relation between objects based on the attribute values. For example, if the information table is incomplete, or when we have an ordered information table.

While in the former case attribute values are missing which makes it impossible to construct an equivalence relation, the equivalence classes in the latter case will mostly consist of only one object which is unreasonable for applications such as rule induction.

In such cases, the family of basic granules for $A \subseteq C$ is not given by a partition, but by a more general covering $\mathbb{C}_A$. A covering $\mathbb{C}$ is a non-empty family of non-empty subsets of U such that $\bigcup \mathbb{C} = U$. Every set or *patch* in $\mathbb{C}_A$ is called an elementary set and should be constructed using meaningful semantics. In addition, every union of patches of $\mathbb{C}_A$ will be interpretable from the data in the information table. Such a union is called a definable set. The granular structure which represents these definable sets is given by

$$\cup^*(\mathbb{C}_A) = \{\bigcup F \mid F \subseteq \mathbb{C}_A\}$$

and is obtained by closing $\mathbb{C}_A$ under set union. It is $\cup$-closed, contains the empty set and its corresponding model of granular structure is denoted by $(U, (\cup^*(\mathbb{C}_A), \cup))$ [28]. Although it is closed under set union, it is not closed under set intersection and set complement such as in the case of Pawlak's rough set model. As every partition can be seen as a covering of the universe, $(U, (\mathscr{B}(U/E_A), \cap, \cup, ^c))$ is a sub-model of $(U, (\cup^*(\mathbb{C}_A), \cup))$.

Similar to the rough set model of Pawlak, a subset $X \subseteq U$ which is not definable can be approximated by definable sets in $\cup^*(\mathbb{C}_A)$. As $\cup^*(\mathbb{C}_A)$ is closed under set union, there exists a unique greatest definable set in $\cup^*(\mathbb{C}_A)$ contained by X, therefore

$$\underline{\text{apr}}_A(X) = \text{the greatest definable set in } \cup^* (\mathbb{C}_A) \text{ contained by } X \tag{1}$$

$$= \bigcup\{Y \in \cup^*(\mathbb{C}_A) \mid Y \subseteq X\}. \tag{2}$$

Unfortunately, as $\cup^*(\mathbb{C}_A)$ is not closed under set intersection, there does not necessarily exist a unique smallest definable set in $\cup^*(\mathbb{C}_A)$ containing X and thus, $\overline{\text{apr}}_A(X)$ is not necessarily an element in $\cup^*(\mathbb{C}_A)$, but a set of minimal elements in $\cup^*(\mathbb{C}_A)$ [28]:

$$\overline{\text{apr}}_A(X) = \{Y \in \cup^*(\mathbb{C}_A) \mid X \subseteq Y, Y \text{ minimal}\},$$

where Y is minimal if $Y \in \cup^*(\mathbb{C}_A)$, $X \subseteq Y$ and $\forall Z \in \cup^*(\mathbb{C}_A)$ with $X \subseteq Z$, if $Z \subseteq Y$ then $Y = Z$. Note that the upper approximation operator of X is given by the definable sets 'just' above X, as they provide the most accurate information.

Hence, as the upper approximation operator is a subset of $\mathscr{P}(U)$ rather than an element of $\mathscr{P}(U)$, various properties from Pawlak's framework no longer make sense, such as the definability of the upper approximation operator and the duality between the lower and upper approximation operator. Note that for $X \in \cup^*(\mathbb{C}_A)$ it does hold that $\underline{\text{apr}}_A(X) = X$ and $\overline{\text{apr}}_A(X) = \{X\}$.

We illustrate the above approximation operators in the following example:

Example 1. [28] Let $U = \{a, b, c, d, e\}$ and $\mathbb{C} = \{\{a\}, \{b, d\}, \{a, b, c\}, \{b, c, e\}\}$, then

$$\cup^*(\mathbb{C}) = \{\emptyset, \{a\}, \{b, d\}, \{a, b, c\}, \{b, c, e\}, \{a, b, d\}, \\ \{a, b, c, d\}, \{a, b, c, e\}, \{b, c, d, e\}, \{a, b, c, d, e\}\}.$$

The lower and upper approximation of $\{a, b, c\}$ are given by $\underline{\text{apr}}(\{a, b, c\}) = \{a, b, c\}$ and $\overline{\text{apr}}(\{a, b, c\}) = \{\{a, b, c\}\}$ and for $\{b, c, d\}$ they are given by $\underline{\text{apr}}(\{b, c, d\}) = \{b, d\}$ and $\overline{\text{apr}}(\{b, c, d\}) = \{\{a, b, c, d\}, \{b, c, d, e\}\}$.

Next, we discuss different characterizations for the lower approximation operator, when a reflexive and transitive neighborhood operator is used instead of an equivalence relation.

3 Different Characterizations for the Lower Approximation Operator Based on a Reflexive and Transitive Neighborhood Operator

As we saw above, the lower approximation operator of Pawlak's model is given by

$$\underline{\text{apr}}(X) = \bigcup\{Y \in \mathscr{B}(U/E) \mid Y \subseteq X\},$$

where E is an equivalence relation based on the data and X is a subset of the universe U. This characterization is called the *subsystem-based* definition of Pawlak's model [27]. Moreover, there are two equivalent characterizations of the lower approximation operator, called the *element-based* and *granule-based* definition, respectively:

$$\underline{\text{apr}}(X) = \{x \in U \mid [x]_E \subseteq X\},$$
$$\underline{\text{apr}}(X) = \bigcup\{[x]_E \mid x \in U, [x]_E \subseteq X\}.$$

In [24], Yao generalized the element-based and granule-based lower approximation operator of Pawlak's model by using neighborhoods instead of equivalence classes. A neighborhood operator $n\colon U \to \mathscr{P}(U)$ maps an object $x \in U$ to a subset $n(x) \subseteq U$. A neighborhood operator n is called reflexive if $x \in n(x)$ for all $x \in U$ and it is called transitive if $x \in n(y)$ implies $n(x) \subseteq n(y)$ for all $x, y \in U$ [24].

Let n be a neighborhood operator and $X \subseteq U$, then the element-based and granule-based lower approximation of X based on n are defined as follows [24]:

$$\underline{\text{apr}}_{1,n}(X) = \{x \in U \mid n(x) \subseteq X\},$$
$$\underline{\text{apr}}_{2,n}(X) = \bigcup\{n(x) \mid x \in U, n(x) \subseteq X\} \\ = \{x \in U \mid \exists y \in U\colon x \in n(y), n(y) \subseteq X\}.$$

In general, the element-based and granule-based definition are no longer equivalent to each other, which was the case in Pawlak's model. However, they are equivalent for a reflexive and transitive neighborhood operator.

Theorem 1. *[24] Let n be a neighborhood operator. The operators $\underline{\text{apr}}_{1,n}$ and $\underline{\text{apr}}_{2,n}$ are equivalent if and only if n is reflexive and transitive.*

Following the discussion in Sect. 2, a meaningful generalization of the subsystem-based definition of Pawlak is provided. The definable sets are now given by the neighborhood system

$$\mathbb{C}_n = \{n(x) \mid x \in U\},$$

which is a covering, instead of the partition U/E. Applying covering $\mathbb{C}_n$ to Eq. (2), we obtain the subsystem-based lower approximation of $X \subseteq U$ based on n:

$$\underline{\text{apr}}_{3,n}(X) = \bigcup\{Y \in \cup^*(\mathbb{C}_n) \mid Y \subseteq X\}.$$

In the following, we discuss that the operator $\underline{\text{apr}}_{3,n}$ is equivalent to the operators $\underline{\text{apr}}_{1,n}$ and $\underline{\text{apr}}_{2,n}$ if n is a reflexive and transitive operator.

Let n be a reflexive and transitive neighborhood operator and let

$$\tau_n = \{X \subseteq U \mid \underline{\text{apr}}_{1,n}(X) = X\}.$$

As n is reflexive, τ_n is a topology [5]. Qin et al. [13] proved the following theorem:

Theorem 2. *[13] If n is a reflexive and transitive operator, then the topology τ_n can be characterized by $\tau_n = \{\underline{\text{apr}}_{1,n}(X) \mid X \subseteq U\}$ and $\underline{\text{apr}}_{1,n}$ is the interior operator of τ_n.*

From this, we derive that for $X \subseteq U$

$$\begin{aligned}\underline{\text{apr}}_{1,n}(X) &= \text{the greatest set in}\, \tau_n \,\text{contained by}\, X\\ &= \bigcup\{Y \in \tau_n \mid Y \subseteq X\},\end{aligned}$$

since τ_n is a topology and $\underline{\text{apr}}_{1,n}$ is its interior operator.

In the following theorem, we prove that all sets in the topology τ_n are definable and that the operators $\underline{\text{apr}}_{1,n}$ and $\underline{\text{apr}}_{3,n}$ are equivalent.

Theorem 3. *Let n be a reflexive and transitive neighborhood operator, then*

1. $\tau_n \subseteq \cup^*(\mathbb{C}_n)$,
2. $\forall X \subseteq U$: $\underline{\text{apr}}_{1,n}(X) = \underline{\text{apr}}_{3,n}(X)$.

Proof. 1. Let $Y \in \tau_n$, then $Y = \underline{\text{apr}}_{1,n}(Y)$. By Theorem 1,

$$Y = \underline{\text{apr}}_{2,n}(Y) = \bigcup\{n(y) \mid y \in U, n(y) \subseteq Y\}.$$

Hence, Y is a union of neighborhood operators from $\mathbb{C}_n$, and therefore, $Y \in \cup^*(\mathbb{C}_n)$.

2. Let $X \subseteq U$. From $\tau_n \subseteq \cup^*(\mathbb{C}_n)$, we immediately derive that $\underline{\text{apr}}_{1,n}(X) \subseteq \underline{\text{apr}}_{3,n}(X)$. On the other hand, let $x \in \underline{\text{apr}}_{3,n}(X)$, then there exists a set $Y \in \cup^*(\mathbb{C}_n)$ such that $x \in Y$ and $Y \subseteq X$. As $Y \in \cup^*(\mathbb{C}_n)$, there is a subset $F \subseteq \mathbb{C}_n$ such that
$$Y = \bigcup\{n(y) \in F\}.$$
Therefore, there exists a neighborhood $n(y) \in F$ such that $x \in n(y)$. As n is reflexive and transitive, we have that $\underline{\text{apr}}_{1,n}(n(y)) = n(y)$ [24]. Hence, $n(y) \in \tau_n$ and since $x \in n(y)$ and $n(y) \subseteq Y \subseteq X$, we derive that $x \in \underline{\text{apr}}_{1,n}(X)$.

From the above theorem, we conclude that the operators $\underline{\text{apr}}_{1,n}$ and $\underline{\text{apr}}_{3,n}$ are equivalent for a reflexive and transitive neighborhood operator. While the latter is constructed from a semantical point of view, the former is preferable from a computational point of view.

In the following section, we discuss how the above semantics and different characterizations of the lower approximation operator can be used to obtain decision rules from a dominance-based rough set model.

4 Decision Rules in a Dominance-Based Rough Set Model

The dominance-based rough set model introduced by Greco et al. [2,3,18] extends the rough set model of Pawlak by using a dominance relation instead of an equivalence relation as indiscernibility relation. A dominance relation is reflexive and transitive and is often called a pre-order. It is preferable to choose a dominance relation instead of an equivalence relation when the domains V_a of the attributes in At are preference-ordered, i.e., if there is a natural order on the possible values of an attribute. A real-life example is the overall evaluation of bank clients based on the evaluations of different risk factors.

Formally, an outranking relation $\succeq_a$ is defined for each attribute $a \in At$ based on the natural order on V_a, i.e., an object $x \in U$ dominates an object $y \in U$, or y is dominated by x, with respect to the attribute a if $I_a(x) \succeq_a I_a(y)$. Such a relation $\succeq_a$ is reflexive and transitive. It is assumed that each relation $\succeq_a$ is complete, i.e., that for every pair of objects one object is dominating the other. This way, we also get preference-ordered decision classes D_i, with $D_i = \{x \in U \mid I_d(x) = i\}$, $i \in V_d$. For $i, j \in V_d$, if $i \succeq_d j$, the objects from D_i are strictly preferred to the objects from D_j. E.g., the bank clients with overall evaluation 'good' are preferable to the clients with overall evaluation 'medium'.

As the decision classes are preference-ordered, we obtain the upward and downward union of classes: for $i \in V_d$ we have
$$D_i^{\geq} = \bigcup\{D_j \in U/d \mid j \geq i\}$$
and
$$D_i^{\leq} = \bigcup\{D_j \in U/d \mid j \leq i\}.$$

An object x belongs to $D_i^{\geq}$ if the decision of x is at least i, while x belongs to $D_i^{\leq}$ if the the decision of x is at most i.

Given a set of conditional attributes $A \subseteq C$, we obtain a relation D_A of U based on A as follows: an object $x \in U$ dominates an object $y \in U$ or y is dominated by x with respect to A if and only if $x \succeq_a y$ for all $a \in A$. The relation D_A is a complete pre-order, since all relations $\succeq_a$ are. Given the relation D_A for $A \subseteq C$ and an object $x \in U$, we can define the A-dominating and A-dominated set of x. The former is given by all predecessors of x by D_A, the latter by the successors of x:

$$D_A^p(x) = \{y \in U \mid yD_A x\}, \tag{3}$$

$$D_A^s(x) = \{y \in U \mid xD_A y\}. \tag{4}$$

An object y belongs to $D_A^p(x)$ if for all attributes $a \in A$ y dominates x with respect to the attribute a, while y belongs to $D_A^s(x)$ if for all attributes $a \in A$ y is dominated by x with respect to the attribute a. Note that both $D_A^p(x)$ and $D_A^s(x)$ are reflexive and transitive neighborhoods of the object x [24]. Moreover, the sets $\mathbb{C}_A^p = \{D_A^p(x) \mid x \in U\}$ and $\mathbb{C}_A^s = \{D_A^s(x) \mid x \in U\}$ are coverings of the universe U. These coverings are meaningful families of basic granules for $A \subseteq C$ as it is clear that every patch $D_A^p(x)$, respectively $D_A^s(x)$, represents the objects which attributes values on A are bounded from below, respectively from above, by the values of x on A. Moreover, the definable sets are given by the $\cup$-closed sets $\cup^*(\mathbb{C}_A^p)$ and $\cup^*(\mathbb{C}_A^s)$. From Sect. 2, the lower approximation of $X \subseteq U$ using $\cup^*(\mathbb{C}_A^p)$ and $\cup^*(\mathbb{C}_A^s)$ is given, respectively, by

$$\underline{\text{apr}}_{3,D_A^p}(X) = \bigcup\{Y \in \cup^*(\mathbb{C}_A^p) \mid Y \subseteq X\}, \tag{5}$$

$$\underline{\text{apr}}_{3,D_A^s}(X) = \bigcup\{Y \in \cup^*(\mathbb{C}_A^s) \mid Y \subseteq X\}, \tag{6}$$

where we inherit the notation for the lower approximation operators from Sect. 3.

To obtain useful knowledge from the decision table, we want to derive decision rules from the given data. More specifically, Greco et al. obtained certain rules from the following lower approximations of the upward and downward union of classes.

$$\underline{\text{apr}}_{1,D_A^p}(D_i^{\geq}) = \{x \in U \mid D_A^p(x) \subseteq D_i^{\geq}\}, \tag{7}$$

$$\underline{\text{apr}}_{1,D_A^s}(D_i^{\leq}) = \{x \in U \mid D_A^s(x) \subseteq D_i^{\leq}\} \tag{8}$$

By Theorem 3, the lower approximations $\underline{\text{apr}}_{1,D_A^p}(D_i^{\geq})$ and $\underline{\text{apr}}_{1,D_A^s}(D_i^{\leq})$ are equal to $\underline{\text{apr}}_{3,D_A^p}(D_i^{\geq})$ and $\underline{\text{apr}}_{3,D_A^s}(D_i^{\leq})$. Hence, the computational approximation operators used by Greco et al. both have a semantically sound counterpart, provided by the framework from Sect. 2.

The interpretation of the lower approximation of an upward union $D_i^{\geq}$ is the following: an object x certainly belongs to $D_i^{\geq}$, i.e., it belongs to its lower approximation, if for every object y which dominates x with respect to A it holds that the decision of y is at least i. Analogously, x certainly belongs to $D_i^{\leq}$ if every object y which is dominated by x with respect to A has a decision at most i. This way, if the evaluation of an object on A improves, the class assignment of the object does not worsen and vice versa, if the evaluation on A is less good, the class assignment does not improve. Therefore, it is not meaningful to consider $\{x \in U \mid D_A^p(x) \subseteq D_i^{\leq}\}$ and $\{x \in U \mid D_A^s(x) \subseteq D_i^{\geq}\}$ as lower approximations, although it can be done from computational point of view.

To obtain certain decision rules, let $A = \{a_1, a_2, \ldots, a_n\} \subseteq C$ and $i \in V_d$. If the lower approximation $\underline{\mathrm{apr}}_{1,D_A^p}(D_i^{\geq})$ is not empty then we derive the certain decision rule

$$\text{if } I_{a_1}(x) \geq v_1 \wedge I_{a_2}(x) \geq v_2 \wedge \ldots \wedge I_{a_n}(x) \geq v_n, \text{ then } I_d(x) \geq i,$$

where $v_i \in V_{a_i}$. Analogously, if $\underline{\mathrm{apr}}_{1,D_A^s}(D_i^{\leq})$ is not empty, then the following certain decision rule is obtained:

$$\text{if } I_{a_1}(x) \leq v_1 \wedge I_{a_2}(x) \leq v_2 \wedge \ldots \wedge I_{a_n}(x) \leq v_n, \text{ then } I_d(x) \leq i.$$

In the above discussion, we only obtained certain decision rules as we only used the lower approximations of the upward and downward unions of decision classes. However, it is also possible to derive possible rules by using the upper approximations, which we obtain as the dual operators from Eqs. (7) and (8):

$$\overline{\mathrm{apr}}_{1,D_A^p}(D_i^{\geq}) = \{x \in U \mid D_A^s(x) \cap D_i^{\geq} \neq \emptyset\}, \tag{9}$$

$$\overline{\mathrm{apr}}_{1,D_A^s}(D_i^{\leq}) = \{x \in U \mid D_A^p(x) \cap D_i^{\leq} \neq \emptyset\}. \tag{10}$$

Note that these upper approximations are obtained from a computational viewpoint, and not from a conceptual one. Although they provide us with possible

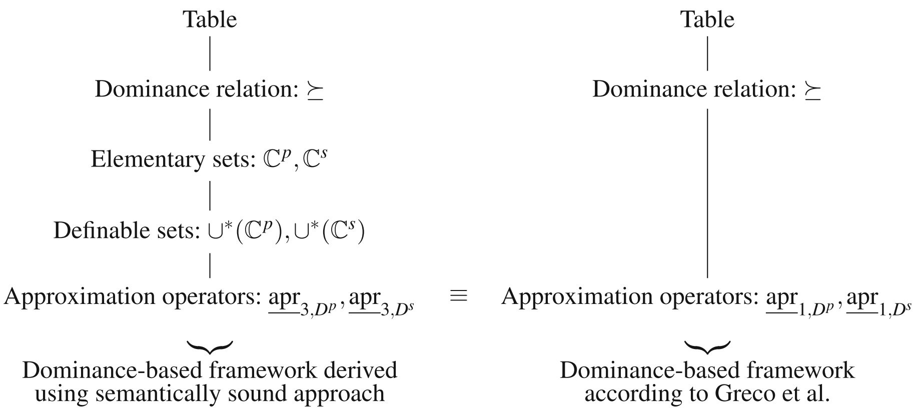

Fig. 1. Comparison between the semantical framework and the dominance-based rough set model

rules which can be used in data analysis, the semantical meaning of these rules is less clear.

To end, we summarize the different steps to obtain the lower approximation operator in the semantically sound approach and the dominance-based rough set approach in Fig. 1. The lower approximation operators from both frameworks are equivalent, but there is no such comparison for the upper approximation operators. By constructing the meaningful coverings $\mathbb{C}^p$ and $\mathbb{C}^s$ via the dominance relation $\succeq$, the dominance-based rough set model can be seen as a special case of the semantically sound framework of rough sets.

5 Conclusion and Future Work

In this paper we have refocussed on the conceptual formulation of rough sets. We discussed a semantical approach of Pawlak's rough set model and covering-based rough set models in which we formalized the elementary and definable sets. Taking the definable sets as primitive notions, meaningful approximation operators are obtained. Unfortunately, since the definable sets in a covering-based rough set model are not closed under set intersection, the upper approximation of a set of the universe in this framework is not a definable set, but it is a set of definable sets.

Furthermore, we have illustrated the semantic approach of covering-based rough sets with the dominance-based rough set model. The obtained conceptual lower approximation operator is in fact equivalent to the known computational lower approximation operator in the dominance-based framework. In addition, we illustrated how to obtain certain decision rules from a preference-ordered decision table.

A future objective is to formalize the elementary and definable sets with respect to a logic language as in [25], by formally describing the intensions and the extensions of the concepts. Moreover, we will study how to obtain a meaningful covering $\mathbb{C}_A$. Furthermore, we want to discuss a semantic approach for covering-based rough sets for an incomplete decision table.

Acknowledgments. Lynn D'eer has been supported by the Ghent University Special Research Fund. This work was partially supported by the Spanish Ministry of Science and Technology under the Project TIN2014-57251-P and the Andalusian Research Plans P10-TIC-6858, P11-TIC-7765 and P12-TIC-2958.

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