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Neighborhood operators for covering-based rough sets

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ABSTRACT

Covering-based rough sets are important generalizations of the classical rough sets of Pawlak. A common way to shape lower and upper approximations within this framework is by means of a neighborhood operator. In this article, we study 24 such neighborhood operators that can be derived from a single covering. We verify equalities between them, reducing the original collection to 13 different neighborhood operators. For the latter, we establish a partial order, showing which operators yield smaller or greater neighborhoods than others.

Six of the considered neighborhood operators result in new covering-based rough set approximation operators. We study how these new approximation operators relate to existing ones in terms of partial order relations, i.e., whether the generated approximations are in general greater, smaller or incomparable. Finally, we discuss the connection between the covering-based approximation operators and relation-based approximation operators, another prominent generalization of Pawlak's rough sets.

1. Introduction

Rough set theory was introduced by Pawlak in 1982, as a tool to deal with uncertainty caused by indiscernibility and incompleteness in information systems [7]. To discern the elements of a universe U, an equivalence relation on U is considered. Pawlak's definition appeared to have many equivalent formulations which are mutually interpretable. As Yao and Yao stated in 2012, the equivalence relation offers two additional equivalent structures [27]: the equivalence classes of the relation form a partition of U, and a σ-algebra is obtained by taking unions of families of the equivalence classes. Yao and Yao refer to the three equivalent definitions as the element-based, the granule-based and the subsystem-based definition, respectively [27]. Moreover, Düntsch and Orłowska [4] considered characterizations through Boolean algebras with operators (Tarski and Jonsson style), while Tang et al. [16] considered matroids.

All three equivalent definitions of Pawlak given by Yao and Yao can be generalized. A first generalization of rough sets is obtained by replacing the equivalence relation by a general binary relation or by a neighborhood operator [14,20,26,34]. In this case, the binary relation or the neighborhood operator determines collections of sets which no longer form a partition of

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U. A second generalization is derived when we substitute the partition obtained by the equivalence relation with a covering; i.e., a collection of non-empty sets such that its union is equal to U [8,26,28]. Finally, to generalize the subsystem-based definition, a closure system over U, i.e., a family of subsets of U that contains U and is closed under set intersection, can be considered [27].

Żakowski proposed the first notion of covering-based rough set approximation operators in 1983 [28]. However, his approximation operators are no longer dual as in Pawlak's case. For this reason, Pomykała [8] studied the operators of Żakowski and their dual operators. Yao [26] studied dual approximation operators by using the predecessor and successor neighborhoods of serial and inverse serial binary relations. While Pomykała and Yao were the founders of the study on dual covering-based rough set approximation operators, Zhu and his co-workers provided many results on non-dual approximation operators [30–37]. Also Tsang et al. [17], Xu and Wang [21] and Xu and Zhang [22] contributed to the study of the non-dual framework. Yang and Li [23] provided a survey on the non-dual framework, in which seven upper approximation operators are distinguished.

A first thorough survey of all dual generalizations of Pawlak's model was done by Samanta and Chakraborty in [12,13] and was further studied by Yao and Yao in 2012 [27], in which 20 pairs of dual approximation operators were considered. In 2014, Restrepo et al. [10] adopted the framework of Yao and Yao, and also integrated the non-dual framework of Yang and Li [23] into it by considering the corresponding dual lower approximation operators. They reduced the number of different covering approximation operators to 16 dual pairs, and studied the partial order relations between these 16 pairs of dual approximation operators [10], showing which operators yield smaller or larger approximations.

In this paper, we focus on the generalization of the element-based definition of Pawlak's model, where the equivalence relation is replaced by a neighborhood operator. We present a detailed study on 24 different neighborhood operators based on a covering, obtained from the cross-linking between four standard neighborhood operators and a covering and its five well-known derivative coverings [27]. Besides studying possible equalities between these neighborhood operators, we discuss their partial order relations, summarized by means of a Hasse diagram. This study is particularly relevant as it shows which neighborhood operators give rise to relatively larger or smaller neighborhoods for each element.

Moreover, our results lead to six new dual covering-based rough approximation operators, which are added to the framework of Restrepo et al. [9,10], and integrated into the Hasse diagrams showing the partial order for lower approximation operators.

In addition, we establish relationships between covering-based approximation operators and relation-based approximation operators, i.e., generalizations of Pawlak's model where a binary relation is used instead of an equivalence relation. Some connections between the two generalizations have already been studied in literature: Zhu established an equivalence between a type of binary relation-based rough sets and a type of covering-based rough sets [34]. Moreover, Zhang and Luo discussed the equivalence between four types of covering-based rough sets and a type of relation-based rough sets, respectively [29]. In this paper, we use a result from [25] to examine which covering-based approximation operators are equivalent with relation-based approximation operators. It will be shown that all covering-based approximation operators equivalent with relation-based operators are related to one of the 24 neighborhood operators. In addition, we obtain the result that studying the properties of the neighborhood operator is sufficient for deriving the properties of the associated binary relation.

The remainder of this paper is structured as follows. In Section 2, we present preliminary concepts on covering-based rough sets and neighborhood operators. Furthermore, we recall the relevant results and definitions stated in [27] and [10]. In Section 3, we introduce 24 neighborhood operators based on coverings, and study equalities and partial order relations that hold for them, in order to obtain the Hasse diagram of the neighborhood operators. In Section 4, we study the new covering-based rough set approximation operators based on neighborhood operators and more specifically, their partial order relations with existing covering-based rough set approximation operators. In Section 5, we discuss the connection of covering-based rough sets with relation-based rough sets. In Section 6, we discuss some ideas concerning applications of covering-based rough sets to data analysis. To end, we state conclusions and outline future work in Section 7. In the Appendix, counterexamples for Section 4.2 are presented.

Finally, we note that this paper extends the conference paper [11], where a limited part of the results we obtain was presented.

2. Preliminaries

Throughout this paper we assume that the universe U is a non-empty set. The original rough set model of Pawlak [7] uses an equivalence relation E to describe indiscernibility between elements of the universe. In this model, a rough set consists of a lower and upper approximation of a set A in U, where the former contains all the elements of U certainly belonging to A and the latter contains the elements possibly belonging to A. More formally, if $[x]_E$ represents the equivalence class of $x \in U$ and $A \subseteq U$, then

$$\underline{\mathrm{apr}}(A) = \{x \in U : [x]_E \subseteq A\} = \bigcup\{[x]_E \in U/E : [x]_E \subseteq A\} \tag{1}$$

is called the lower approximation of A and

$$\overline{\mathrm{apr}}(A) = \{x \in U : [x]_E \cap A \neq \emptyset\} = \bigcup\{[x]_E \in U/E : [x]_E \cap A \neq \emptyset\} \tag{2}$$

is called the upper approximation of A. The quotient set U/E represents the set of equivalence classes defined from the equivalence relation E. The first equality in Eqs. (1) and (2) is sometimes called the element-based definition and the second one is called the granule-based definition of Pawlak's model [27].

By weakening the condition of an equivalence relation, many generalizations of Pawlak's model can be defined. An important generalization can be obtained by replacing the partition U/E with a covering of U.

Definition 1. [30] Let $\mathbb{C} = \{K_i \subseteq U : i \in I\}$ be a family of non-empty subsets of U, with I a set of indices. $\mathbb{C}$ is called a covering of U if $\bigcup\limits_{i \in I} K_i = U$. The ordered pair $(U, \mathbb{C})$ is called a covering approximation space.

It is clear that a partition generated by an equivalence relation is a special case of a covering of U. In a covering approximation space, equivalence classes can be generalized to neighborhood operators.

Definition 2. [27] A neighborhood operator is a mapping $N : U \to \mathcal{P}(U)$, where $\mathcal{P}(U)$ represents the collection of subsets of U.

In general, it is assumed that a neighborhood operator N is reflexive, i.e., $\forall x \in U$: $x \in N(x)$, in order to fulfil the intuitive idea of a *neighborhood*.

Each neighborhood operator N defines an ordered pair $(\underline{\text{apr}}_N, \overline{\text{apr}}_N)$ of element-based approximation operators with

$$\underline{\text{apr}}_N(A) = \{x \in U : N(x) \subseteq A\} \tag{3}$$

and

$$\overline{\text{apr}}_N(A) = \{x \in U : N(x) \cap A \neq \emptyset\} \tag{4}$$

for $A \subseteq U$. The approximation operators $\underline{\text{apr}}_N$ and $\overline{\text{apr}}_N$ are dual to each other, i.e.,

$$\forall A \subseteq U : \underline{\text{apr}}_N(\text{co}\, A) = \text{co}(\overline{\text{apr}}_N(A)),$$

where co represents the set-theoretical complement.

Different neighborhood operators, and hence different element-based definitions of covering-based rough sets, can be obtained from a covering $\mathbb{C}$. In general, we are interested in the sets K in $\mathbb{C}$ such that $x \in K$:

Definition 3. [27] If $\mathbb{C}$ is a covering of U and $x \in U$, the neighborhood system $\mathcal{C}(\mathbb{C}, x)$ of x is defined by

$$\mathcal{C}(\mathbb{C}, x) = \{K \in \mathbb{C} : x \in K\}. \tag{5}$$

In a neighborhood system $\mathcal{C}(\mathbb{C}, x)$, the minimal and maximal sets that contain an element $x \in U$ are particularly important.

Definition 4. Let $(U, \mathbb{C})$ be a covering approximation space and x in U. The set

$$\text{md}(\mathbb{C}, x) = \{K \in \mathcal{C}(\mathbb{C}, x) : (\forall S \in \mathcal{C}(\mathbb{C}, x))(S \subseteq K \Rightarrow K = S)\} \tag{6}$$

is called the minimal description [2] of x. On the other hand, the set

$$\text{MD}(\mathbb{C}, x) = \{K \in \mathcal{C}(\mathbb{C}, x) : (\forall S \in \mathcal{C}(\mathbb{C}, x))(S \supseteq K \Rightarrow K = S)\} \tag{7}$$

is called the maximal description [35] of x.

The sets $\text{md}(\mathbb{C}, x)$ and $\text{MD}(\mathbb{C}, x)$ are also called the minimal-description and maximal-description neighborhood systems of x [27]. The importance of the minimal and maximal description of x is demonstrated by the following proposition:

Proposition 1. *[27] Let $\mathbb{C}$ be a covering and $K \in \mathcal{C}(\mathbb{C}, x)$. Then there exist $K_1 \in \text{md}(\mathbb{C}, x)$ and $K_2 \in \text{MD}(\mathbb{C}, x)$ such that $K_1 \subseteq K \subseteq K_2$. Moreover, for all $x \in U$ it holds that $\bigcap \text{md}(\mathbb{C}, x) = \bigcap \mathcal{C}(\mathbb{C}, x)$ and $\bigcup \text{MD}(\mathbb{C}, x) = \bigcup \mathcal{C}(\mathbb{C}, x)$.*

Besides the neighborhood system $\mathcal{C}(\mathbb{C}, x)$, we can consider the two extreme neighborhood systems of an element $x \in U$, namely $\text{md}(\mathbb{C}, x)$ and $\text{MD}(\mathbb{C}, x)$. Furthermore, we can apply two extreme operations on these neighborhood systems by taking the intersection and the union. Therefore, Yao and Yao [27] constructed the following four neighborhood operators based on the covering $\mathbb{C}$:

1. $N_1^{\mathbb{C}}(x) = \bigcap\{K \in \mathbb{C} : K \in \text{md}(\mathbb{C}, x)\} = \bigcap \mathcal{C}(\mathbb{C}, x)$,
2. $N_2^{\mathbb{C}}(x) = \bigcup\{K \in \mathbb{C} : K \in \text{md}(\mathbb{C}, x)\}$,
3. $N_3^{\mathbb{C}}(x) = \bigcap\{K \in \mathbb{C} : K \in \text{MD}(\mathbb{C}, x)\}$,
4. $N_4^{\mathbb{C}}(x) = \bigcup\{K \in \mathbb{C} : K \in \text{MD}(\mathbb{C}, x)\} = \bigcup \mathcal{C}(\mathbb{C}, x)$.

Therefore, for each $N_i^{\mathbb{C}}$ with $i = 1, 2, 3, 4$, we have a pair of dual approximation operators $(\underline{\text{apr}}_{N_i}^{\mathbb{C}}, \overline{\text{apr}}_{N_i}^{\mathbb{C}})$ defined in Eqs. (3) and (4).

On the other hand, Yao and Yao [27] also considered six coverings derived from a covering $\mathbb{C}$

1. $\mathbb{C}_1 = \bigcup\{\text{md}(\mathbb{C}, x) : x \in U\}$,

2. $\mathbb{C}_2 = \bigcup\{\mathrm{MD}(\mathbb{C}, x) : x \in U\}$,
3. $\mathbb{C}_3 = \{\bigcap \mathrm{md}(\mathbb{C}, x) : x \in U\} = \{\bigcap \mathcal{C}(\mathbb{C}, x) : x \in U\}$,
4. $\mathbb{C}_4 = \{\bigcup \mathrm{MD}(\mathbb{C}, x) : x \in U\} = \{\bigcup \mathcal{C}(\mathbb{C}, x) : x \in U\}$,
5. $\mathbb{C}_\cap = \mathbb{C} \setminus \left\{K \in \mathbb{C} : \left(\exists \mathbb{C}' \subseteq \mathbb{C} \setminus \{K\}\right)\left(K = \bigcap \mathbb{C}'\right)\right\}$,
6. $\mathbb{C}_\cup = \mathbb{C} \setminus \left\{K \in \mathbb{C} : \left(\exists \mathbb{C}' \subseteq \mathbb{C} \setminus \{K\}\right)\left(K = \bigcup \mathbb{C}'\right)\right\}$.

The idea behind the first two coverings is similar to the rationale for $N_1^{\mathbb{C}}$, $N_2^{\mathbb{C}}$, $N_3^{\mathbb{C}}$ and $N_4^{\mathbb{C}}$. Given the extreme neighborhood systems $\mathrm{md}(\mathbb{C}, x)$ and $\mathrm{MD}(\mathbb{C}, x)$ for $x \in U$, then the union of these systems leads to new coverings. Note that this is not the case when taking the intersection. Coverings $\mathbb{C}_3$ and $\mathbb{C}_4$ are directly related with $N_1^{\mathbb{C}}$ and $N_4^{\mathbb{C}}$. Covering $\mathbb{C}_\cap$ is called the intersection reduct and $\mathbb{C}_\cup$ the union reduct. These reducts eliminate intersection reducible elements, resp. union reducible elements, from the covering, respectively. An intersection reducible element of a covering $\mathbb{C}$ is an element $K \in \mathbb{C}$ such that $\exists \mathbb{C}' \subseteq \mathbb{C} \setminus \{K\} : K = \bigcap \mathbb{C}'$, while a union reducible element of $\mathbb{C}$ is an element $K \in \mathbb{C}$ such that $\exists \mathbb{C}' \subseteq \mathbb{C} \setminus \{K\} : K = \bigcup \mathbb{C}'$.

The equality $\mathbb{C}_1 = \mathbb{C}_\cup$ was established in [10], while the other coverings are different in general. Also, note that $\mathbb{C}_1$, $\mathbb{C}_2$ and $\mathbb{C}_\cap$ are subcoverings of $\mathbb{C}$, while $\mathbb{C}_3$ and $\mathbb{C}_4$ are not.

To end this section, we discuss the models relevant for this paper from the framework of covering-based rough sets which was introduced by Yao and Yao in [27]. First, we discuss the granule-based definitions which generalize the second equalities in Eqs. (1) and (2).

Definition 5. [27] Let $\mathbb{C}$ be a covering, then we have two pairs of dual approximation operators $(\underline{\mathrm{apr}}'_{\mathbb{C}}, \overline{\mathrm{apr}}'_{\mathbb{C}})$ and $(\underline{\mathrm{apr}}''_{\mathbb{C}}, \overline{\mathrm{apr}}''_{\mathbb{C}})$ which are defined, for $A \subseteq U$,

$$\underline{\mathrm{apr}}'_{\mathbb{C}}(A) = \bigcup\{K \in \mathbb{C} : K \subseteq A\}, \tag{8}$$

$$\overline{\mathrm{apr}}'_{\mathbb{C}}(A) = \mathrm{co}(\underline{\mathrm{apr}}'_{\mathbb{C}}(\mathrm{co}(A))), \tag{9}$$

$$\underline{\mathrm{apr}}''_{\mathbb{C}}(A) = \mathrm{co}(\overline{\mathrm{apr}}''_{\mathbb{C}}(\mathrm{co}(A))), \tag{10}$$

$$\overline{\mathrm{apr}}''_{\mathbb{C}}(A) = \bigcup\{K \in \mathbb{C} : K \cap A \neq \emptyset\}. \tag{11}$$

Second, we recall the definition of system-based approximation operators. Yao and Yao used the notion of a closure system over U, i.e., a family of subsets of U that contains U and is closed under set intersection [27]. Given a closure system $\mathbb{S}$ over U, one can construct its dual system $\mathbb{S}' = \{\mathrm{co}(K) : K \in \mathbb{S}\}$. The system $\mathbb{S}'$ contains $\emptyset$ and is closed under set union.

Definition 6. [27] Given $S = (\mathbb{S}', \mathbb{S})$, a pair of dual lower and upper approximations can be defined as follows: for $A \subseteq U$,

$$\underline{\mathrm{apr}}_S(A) = \bigcup\{K \in \mathbb{S}' : K \subseteq A\}, \tag{12}$$

$$\overline{\mathrm{apr}}_S(A) = \bigcap\{K \in \mathbb{S} : K \supseteq A\}. \tag{13}$$

As a particular example of a closure system, [27] considered the intersection closure $S_{\cap,\mathbb{C}}$ of a covering $\mathbb{C}$, i.e., the minimal subset of $\mathcal{P}(U)$ that contains $\mathbb{C}$, $\emptyset$ and U, and is closed under set intersection. On the other hand, the union closure of $\mathbb{C}$, denoted by $S_{\cup,\mathbb{C}}$, is the minimal subset of $\mathcal{P}(U)$ that contains $\mathbb{C}$, $\emptyset$ and U, and is closed under set union. It can be shown that the dual system $(S_{\cup,\mathbb{C}})'$ forms a closure system. Both $S_\cap = ((S_{\cap,\mathbb{C}})', S_{\cap,\mathbb{C}})$ and $S_\cup = (S_{\cup,\mathbb{C}}, (S_{\cup,\mathbb{C}})')$ can be used to obtain two pairs of dual approximation operations, according to Eqs. (12) and (13).

Furthermore, various authors [8,17,21,22,28,34,36] have studied non-dual pairs of approximation operators. Yang and Li [23] provided an overview of these non-dual operators. Restrepo et al. [10] studied the relationship between these approximation operators and the framework from [27] and found that some of them coincide. The upper approximation operators stated in [23] which do not correspond to any operator studied in [27] are listed below[1].

Definition 7. [23] Let $\mathbb{C}$ be a covering, then we can define four upper approximation operators $H : \mathcal{P}(U) \to \mathcal{P}(U)$ as follows: let $A \subseteq U$, then

$$H_1^{\mathbb{C}}(A) = \underline{\mathrm{apr}}'_{\mathbb{C}}(A) \cup \left(\bigcup\left\{\bigcup \mathrm{md}(\mathbb{C}, x) : x \in A \setminus \underline{\mathrm{apr}}'_{\mathbb{C}}(A)\right\}\right), \tag{14}$$

$$H_3^{\mathbb{C}}(A) = \bigcup\left\{\bigcup \mathrm{md}(\mathbb{C}, x) : x \in A\right\} = \bigcup\{N_2^{\mathbb{C}}(x) : x \in A\}, \tag{15}$$

$$H_4^{\mathbb{C}}(A) = \underline{\mathrm{apr}}'_{\mathbb{C}}(A) \cup \left(\bigcup\left\{K \in \mathbb{C} : K \cap (A \setminus \underline{\mathrm{apr}}'_{\mathbb{C}}(A)) \neq \emptyset\right\}\right), \tag{16}$$

$$H_5^{\mathbb{C}}(A) = \bigcup\{N_1^{\mathbb{C}}(x) : x \in A\}. \tag{17}$$

[1] Note that our definition of $H_1^{\mathbb{C}}$ and $H_3^{\mathbb{C}}$ slightly differs from the one presented in [23]: indeed, we need to add one extra $\bigcup$-symbol compared to [23], since the minimal description of x is a set of sets.

Table 1
Framework of 16 dual pairs of approximation operators.

Number	Lower approximation	Upper approximation
1	$\underline{\mathrm{apr}}^{\mathbb{C}}_{N_1} = \underline{\mathrm{apr}}'_{\mathbb{C}_3}$	$\overline{\mathrm{apr}}^{\mathbb{C}}_{N_1} = \overline{\mathrm{apr}}'_{\mathbb{C}_3}$
2	$\underline{\mathrm{apr}}^{\mathbb{C}}_{N_2}$	$\overline{\mathrm{apr}}^{\mathbb{C}}_{N_2}$
3	$\underline{\mathrm{apr}}^{\mathbb{C}}_{N_3}$	$\overline{\mathrm{apr}}^{\mathbb{C}}_{N_3}$
4	$\underline{\mathrm{apr}}^{\mathbb{C}}_{N_4} = \underline{\mathrm{apr}}''_{\mathbb{C}} = \underline{\mathrm{apr}}''_{\mathbb{C}_2} = \underline{\mathrm{apr}}''_{\mathbb{C}_\cap}$	$\overline{\mathrm{apr}}^{\mathbb{C}}_{N_4} = \overline{\mathrm{apr}}''_{\mathbb{C}} = \overline{\mathrm{apr}}''_{\mathbb{C}_2} = \overline{\mathrm{apr}}''_{\mathbb{C}_\cap}$
5	$\underline{\mathrm{apr}}'_{\mathbb{C}} = \underline{\mathrm{apr}}'_{\mathbb{C}_1} = \underline{\mathrm{apr}}_{S_\cup}$	$\overline{\mathrm{apr}}'_{\mathbb{C}} = \overline{\mathrm{apr}}'_{\mathbb{C}_1} = \overline{\mathrm{apr}}_{S_\cup}$
6	$\underline{\mathrm{apr}}'_{\mathbb{C}_2}$	$\overline{\mathrm{apr}}'_{\mathbb{C}_2}$
7	$\underline{\mathrm{apr}}'_{\mathbb{C}_4}$	$\overline{\mathrm{apr}}'_{\mathbb{C}_4}$
8	$\underline{\mathrm{apr}}'_{\mathbb{C}_\cap}$	$\overline{\mathrm{apr}}'_{\mathbb{C}_\cap}$
9	$\underline{\mathrm{apr}}''_{\mathbb{C}_1}$	$\overline{\mathrm{apr}}''_{\mathbb{C}_1}$
10	$\underline{\mathrm{apr}}''_{\mathbb{C}_3}$	$\overline{\mathrm{apr}}''_{\mathbb{C}_3}$
11	$\underline{\mathrm{apr}}''_{\mathbb{C}_4}$	$\overline{\mathrm{apr}}''_{\mathbb{C}_4}$
12	$\underline{\mathrm{apr}}_{S_\cap}$	$\overline{\mathrm{apr}}_{S_\cap}$
13	$(H_1^{\mathbb{C}})^\partial$	$H_1^{\mathbb{C}}$
14	$(H_3^{\mathbb{C}})^\partial$	$H_3^{\mathbb{C}}$
15	$(H_4^{\mathbb{C}})^\partial$	$H_4^{\mathbb{C}}$
16	$(H_5^{\mathbb{C}})^\partial$	$H_5^{\mathbb{C}}$

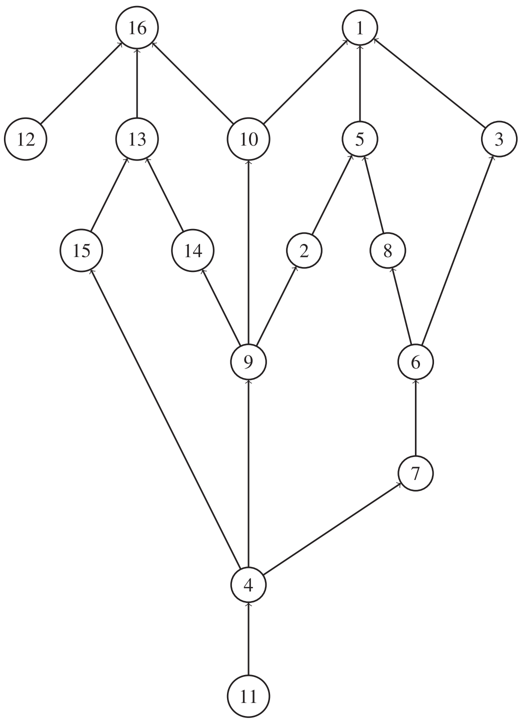

Fig. 1. Hasse diagram for lower approximation operators from Table 1.

The dual lower approximation operators of $H_1^{\mathbb{C}}$, $H_3^{\mathbb{C}}$, $H_4^{\mathbb{C}}$ and $H_5^{\mathbb{C}}$ are denoted by $(H_1^{\mathbb{C}})^\partial$, $(H_3^{\mathbb{C}})^\partial$, $(H_4^{\mathbb{C}})^\partial$ and $(H_5^{\mathbb{C}})^\partial$, respectively, where $(H_i^{\mathbb{C}})^\partial(A) = \mathrm{co}(H_i^{\mathbb{C}}(\mathrm{co}(A)))$.

To summarize, we recall the framework of 16 different dual pairs of approximation operators considered in [9] in Table 1. In [10], Restrepo et al. studied the partial order relation between these 16 pairs. Here, the order relation $\leq$ for two lower approximation operators $\underline{\mathrm{apr}}_1$ and $\underline{\mathrm{apr}}_2$ is defined as follows:

$$\underline{\mathrm{apr}}_1 \leq \underline{\mathrm{apr}}_2 \Leftrightarrow \forall A \subseteq U : \underline{\mathrm{apr}}_1(A) \subseteq \underline{\mathrm{apr}}_2(A). \tag{18}$$

Since all considered pairs are dual, we have the following partial order for the upper approximation operators:

$$\forall A \subseteq U : \overline{\mathrm{apr}}_1(A) \subseteq \overline{\mathrm{apr}}_2(A) \Leftrightarrow \underline{\mathrm{apr}}_2(A) \subseteq \underline{\mathrm{apr}}_1(A).$$

The Hasse diagram representing the partial order of the 16 lower approximation operators is shown in Fig. 1. A Hasse diagram is a mathematical diagram which represents a finite partially ordered set $(P, \leq)$: each element $p \in P$ is represented by a vertex, and there is a directed edge from the element p to the element q if $p \leq q$ and if there is no $r \in P\backslash\{p, q\}$ with $p \leq r \leq q$. From the Hasse diagram, we learn for example that $\underline{\text{apr}}''_{\mathbb{C}_4}$ (operator 11) is smaller or equal than all remaining lower approximation operators, except for $\underline{\text{apr}}_{S_\cap}$ (operator 12), to which it is incomparable. On the other hand, $\underline{\text{apr}}^{\mathbb{C}}_{N_1}$ (operator 1) and $(H_5^{\mathbb{C}})^\partial$ (operator 16) are not dominated by any lower approximation operator in the framework.

3. Neighborhood operators based on coverings

In the framework of Yao and Yao [27], the neighborhood operators $N_i^{\mathbb{C}}$ and the corresponding element-based approximation operators $(\underline{\text{apr}}^{\mathbb{C}}_{N_i}, \overline{\text{apr}}^{\mathbb{C}}_{N_i})$ are considered only for the original covering $\mathbb{C}$. An interesting question, therefore, is what happens when neighborhoods are generated using one of the derived coverings $\mathbb{C}_1$–$\mathbb{C}_4$ and $\mathbb{C}_\cap$. For example, let us consider the combination of neighborhood operator N_1 and covering $\mathbb{C}_4$: let $x, y \in U$, then

$$\begin{aligned} y \in N_1^{\mathbb{C}_4}(x) &\Leftrightarrow \forall K \in \mathcal{C}(\mathbb{C}_4, x) : y \in K \\ &\Leftrightarrow \forall z \in U : x \in \bigcup \mathcal{C}(\mathbb{C}, z) \Rightarrow y \in \bigcup \mathcal{C}(\mathbb{C}, z) \\ &\Leftrightarrow \forall z \in U : z \in \bigcup \mathcal{C}(\mathbb{C}, x) \Rightarrow z \in \bigcup \mathcal{C}(\mathbb{C}, y) \\ &\Leftrightarrow \bigcup \mathcal{C}(\mathbb{C}, x) \subseteq \bigcup \mathcal{C}(\mathbb{C}, y) \\ &\Leftrightarrow N_4^{\mathbb{C}}(x) \subseteq N_4^{\mathbb{C}}(y). \end{aligned}$$

In other words, an element y belongs to $N_1^{\mathbb{C}_4}(x)$ if all elements associated with x are also associated with y.

By combining the four neighborhood operators and six coverings, we obtain 24 neighborhood operators $N_i^{\mathbb{C}_j}$, with $N_i \in \{N_1, N_2, N_3, N_4\}$ and $\mathbb{C}_j \in \{\mathbb{C}, \mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4, \mathbb{C}_\cap\}$. In this paper, we establish equality and inclusion relations between these operators. In Section 3.1, we first study whether the 24 neighborhood operators $N_i^{\mathbb{C}_j}$ are all unique. Next, in Section 3.2, we establish the Hasse diagram formed by the operators.

3.1. Equalities between neighborhood operators

In order to study possible equalities between different neighborhood operators, we first show the following results for the minimal and maximal descriptions of some of the derived coverings.

Proposition 2. *Let $\mathbb{C}$ be a covering and $x \in U$, then*

(a) $\text{md}(\mathbb{C}_1, x) = \text{md}(\mathbb{C}, x)$,
(b) $\text{md}(\mathbb{C}_2, x) = \mathcal{C}(\mathbb{C}_2, x) = \text{MD}(\mathbb{C}_2, x) = \text{MD}(\mathbb{C}, x)$,
(c) $\text{md}(\mathbb{C}_3, x) = \left\{\bigcap \text{md}(\mathbb{C}, x)\right\} = \left\{\bigcap \mathcal{C}(\mathbb{C}, x)\right\}$,
(d) $\bigcap \text{md}(\mathbb{C}_\cap, x) = \bigcap \mathcal{C}(\mathbb{C}_\cap, x) = \bigcap \mathcal{C}(\mathbb{C}, x) = \bigcap \text{md}(\mathbb{C}, x)$,
(e) $\text{MD}(\mathbb{C}_\cap, x) = \text{MD}(\mathbb{C}, x)$.

Proof.

(a) Take $x \in U$, we will prove that $\text{md}(\mathbb{C}_1, x) = \text{md}(\mathbb{C}, x)$.
Let $K \in \text{md}(\mathbb{C}, x)$, then by definition it holds that $K \in \mathbb{C}_1$ and $x \in K$. We will prove that $K \in \text{md}(\mathbb{C}_1, x)$: let $K' \in \mathbb{C}_1$ with $x \in K'$ and $K' \subseteq K$, then $K' \in \mathbb{C}$ since $\mathbb{C}_1 \subseteq \mathbb{C}$. Because $K \in \text{md}(\mathbb{C}, x)$, we have that $K = K'$ and hence, $K \in \text{md}(\mathbb{C}_1, x)$. Therefore, $\text{md}(\mathbb{C}, x) \subseteq \text{md}(\mathbb{C}_1, x)$.
On the other hand, let $K \in \text{md}(\mathbb{C}_1, x)$, then $K \in \mathbb{C}_1$ and $x \in K$. Since $\mathbb{C}_1 \subseteq \mathbb{C}$, $K \in \mathbb{C}$ and since $x \in K$, there exists a $K' \in \text{md}(\mathbb{C}, x)$ with $K' \subseteq K$. Hence, $K' \in \mathbb{C}_1$ and since $K \in \text{md}(\mathbb{C}_1, x)$, we have that $K = K'$. Therefore, $K \in \text{md}(\mathbb{C}, x)$ and $\text{md}(\mathbb{C}_1, x) \subseteq \text{md}(\mathbb{C}, x)$.
We conclude that $\text{md}(\mathbb{C}_1, x) = \text{md}(\mathbb{C}, x)$.

(b)
- Take $x \in U$. We will prove that $\text{md}(\mathbb{C}_2, x) = \mathcal{C}(\mathbb{C}_2, x)$.
By definition, $\text{md}(\mathbb{C}_2, x) \subseteq \mathcal{C}(\mathbb{C}_2, x)$.
On the other hand, take $K \in \mathcal{C}(\mathbb{C}_2, x)$ and $K' \in \mathbb{C}_2$ with $x \in K'$ and $K' \subseteq K$. Since $K' \in \mathbb{C}_2$, there exists a $y \in U$ such that $K' \in \text{MD}(\mathbb{C}, y)$. Since $K' \subseteq K$, $y \in K$ and since $\mathbb{C}_2 \subseteq \mathbb{C}$, $K \in \mathbb{C}$, thus $K \in \mathcal{C}(\mathbb{C}, y)$. Since $K' \in \text{MD}(\mathbb{C}, y)$ and $K' \subseteq K$, we obtain that $K' = K$. Hence, $K \in \text{md}(\mathbb{C}_2, x)$.
We conclude that $\text{md}(\mathbb{C}_2, x) = \mathcal{C}(\mathbb{C}_2, x)$.
- Take $x \in U$. We will prove that $\mathcal{C}(\mathbb{C}_2, x) = \text{MD}(\mathbb{C}_2, x)$.
By definition, $\text{MD}(\mathbb{C}_2, x) \subseteq \mathcal{C}(\mathbb{C}_2, x)$.
On the other hand, take $K \in \mathcal{C}(\mathbb{C}_2, x)$ and $K' \in \mathbb{C}_2$ with $x \in K'$ and $K \subseteq K'$. Since $K \in \mathbb{C}_2$, there exists a $y \in U$ such that $K \in \text{MD}(\mathbb{C}, y)$. Since $K \subseteq K'$, $y \in K'$ and since $\mathbb{C}_2 \subseteq \mathbb{C}$, $K' \in \mathbb{C}$, thus $K' \in \mathcal{C}(\mathbb{C}, y)$. Since $K \in \text{MD}(\mathbb{C}, y)$ and $K \subseteq K'$, we obtain that $K = K'$. Hence, $K \in \text{MD}(\mathbb{C}_2, x)$.
We conclude that $\mathcal{C}(\mathbb{C}_2, x) = \text{MD}(\mathbb{C}_2, x)$.

- Take $x \in U$. We will prove that $\mathrm{MD}(\mathbb{C}_2, x) = \mathrm{MD}(\mathbb{C}, x)$.
  Let $K \in \mathrm{MD}(\mathbb{C}, x)$, then by definition it holds that $K \in \mathbb{C}_2$ and $x \in K$. We will prove that $K \in \mathrm{MD}(\mathbb{C}_2, x)$: let $K' \in \mathbb{C}_2$ with $x \in K'$ and $K \subseteq K'$, then $K' \in \mathbb{C}$ since $\mathbb{C}_2 \subseteq \mathbb{C}$. Because $K \in \mathrm{MD}(\mathbb{C}, x)$, we have that $K = K'$ and hence, $K \in \mathrm{MD}(\mathbb{C}_2, x)$. Therefore, $\mathrm{MD}(\mathbb{C}, x) \subseteq \mathrm{MD}(\mathbb{C}_2, x)$.
  On the other hand, let $K \in \mathrm{MD}(\mathbb{C}_2, x)$, then $K \in \mathbb{C}_2$ and $x \in K$. Since $\mathbb{C}_2 \subseteq \mathbb{C}$, $K \in \mathbb{C}$ and since $x \in K$, there exists a $K' \in \mathrm{MD}(\mathbb{C}, x)$ with $K \subseteq K'$. Hence, $K' \in \mathbb{C}_2$ and since $K \in \mathrm{MD}(\mathbb{C}_2, x)$, we have that $K = K'$. Therefore, $K \in \mathrm{MD}(\mathbb{C}, x)$ and $\mathrm{MD}(\mathbb{C}_2, x) \subseteq \mathrm{MD}(\mathbb{C}, x)$.
  We conclude that $\mathrm{MD}(\mathbb{C}_2, x) = \mathrm{MD}(\mathbb{C}, x)$.

(c) Take $x \in U$ and $K \in \mathrm{md}(\mathbb{C}_3, x)$. Since $K \in \mathbb{C}_3$, there exist $z \in U$ such that $K = \bigcap \mathcal{C}(\mathbb{C}, z)$. Denote $L = \bigcap \mathcal{C}(\mathbb{C}, x)$, so $L \in \mathbb{C}_3$. We will prove that $K = L$.
Take $y \in L$, then for all $M \in \mathbb{C}$ it holds that if $x \in M$ then $y \in M$. Furthermore, since $x \in K$ it holds for all $M \in \mathbb{C}$ that if $z \in M$ then $x \in M$. Hence, for all $M \in \mathbb{C}$ it holds that if $z \in M$ then $y \in M$, thus, $y \in K$. Hence, $L \subseteq K$ and since $K \in \mathrm{md}(\mathbb{C}_3, x)$ and $L \in \mathbb{C}_3$ with $x \in L$, we conclude that $K = L$.

(d) Take $x \in U$. We will prove that $\bigcap \mathcal{C}(\mathbb{C}_\cap, x) = \bigcap \mathcal{C}(\mathbb{C}, x)$.
Since $\mathbb{C}_\cap \subseteq \mathbb{C}$, we always have $\bigcap \mathcal{C}(\mathbb{C}_\cap, x) \supseteq \bigcap \mathcal{C}(\mathbb{C}, x)$.
For the other inclusion, if $K \in \mathcal{C}(\mathbb{C}, x) \setminus \mathcal{C}(\mathbb{C}_\cap, x)$, then there exist $\{K_i : i \in I\} \subseteq \mathbb{C}$ such that $K_i \neq K$ for all i and

$$K = \bigcap_{i \in I} K_i.$$

We can assume that the K_i's are in $\mathbb{C}_\cap$, otherwise we decompose K_i itself into elements of $\mathbb{C}_\cap$. Moreover, since $x \in K$, $x \in K_i$ for $i \in I$.
Let $y \in \bigcap \mathcal{C}(\mathbb{C}_\cap, x) \setminus \bigcap \mathcal{C}(\mathbb{C}, x)$, then for all $K \in \mathbb{C}_\cap$ it holds that if $x \in K$ then $y \in K$ and there exists a $K^* \in \mathbb{C}$ such that $x \in K^*$ and $y \notin K^*$. Hence, there exists a $K^* \in \mathcal{C}(\mathbb{C}, x) \setminus \mathcal{C}(\mathbb{C}_\cap, x)$ with $y \notin K^*$. We can decompose K^* into elements of $\mathbb{C}_\cap$ as we saw before: $K^* = \bigcap_{i \in I} K_i$ with $K_i \neq K^*$. Since $y \notin K^*$, there exists a K_i with $K_i \in \mathbb{C}_\cap$, $x \in K_i$ and $y \notin K_i$, which is a contradiction.
We conclude that $\bigcap \mathcal{C}(\mathbb{C}_\cap, x) = \bigcap \mathcal{C}(\mathbb{C}, x)$.

(e) Take $x \in U$. We will prove that $\mathrm{MD}(\mathbb{C}, x) = \mathrm{MD}(\mathbb{C}_\cap, x)$.
First, let us consider $K \in \mathrm{MD}(\mathbb{C}, x)$ and $K' \in \mathbb{C}_\cap$ with $x \in K'$ and $K \subseteq K'$. Since $K' \in \mathbb{C}$ and $K \in \mathrm{MD}(\mathbb{C}, x)$, we have that $K = K'$ and thus $K \in \mathbb{C}_\cap$ and $K \in \mathrm{MD}(\mathbb{C}_\cap, x)$.
On the other hand, take $K \in \mathrm{MD}(\mathbb{C}_\cap, x)$ and $K' \in \mathbb{C}$ with $x \in K'$ and $K \subseteq K'$. If $K' \in \mathbb{C}_\cap$, then $K = K'$. If $K' \notin \mathbb{C}_\cap$, then there exist $\{K_i : i \in I\} \subseteq \mathbb{C}_\cap \setminus \{K'\}$ with $K' = \bigcap_{i \in I} K_i$. Then for all i, $K \subseteq K_i$ and thus, $K = K_i$ for all i. Again we can conclude that $K = K'$. Hence, $K \in \mathrm{MD}(\mathbb{C}, x)$.
We conclude that $\mathrm{MD}(\mathbb{C}, x) = \mathrm{MD}(\mathbb{C}_\cap, x)$.

□

From Proposition 2, we derive some more insight into the construction of the different coverings. First, the covering $\mathbb{C}_1$ preserves the minimal description of all the elements. Next, we obtain that the minimal and maximal description in $\mathbb{C}_2$ of an element x correspond to all the sets in $\mathbb{C}_2$ that contain x. Moreover, these sets are exactly the sets of the maximal description in $\mathbb{C}$ of x. Furthermore, we derive that the minimal description of x by $\mathbb{C}_3$ is the singleton $\left\{\bigcap \mathcal{C}(\mathbb{C}, x)\right\}$ which corresponds to the intersection of all elements in $\mathcal{C}(\mathbb{C}, x)$. Next, for $\mathbb{C}_\cap$, we obtain that it preserves the intersection of the sets in $\mathbb{C}$ which contain an arbitrary element x, and this for all elements $x \in U$. This is in line with the idea of $\mathbb{C}_\cap$, i.e., $\mathbb{C}_\cap$ omits intersection reducible sets from $\mathbb{C}$. Moreover, $\mathbb{C}_\cap$ preserves the maximal description in $\mathbb{C}$ of all the elements.

We continue with establishing equalities between neighborhood operators by considering the following example:

Example 1. For the covering $\mathbb{C} = \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}\}$ of $U = \{1, 2, 3, 4\}$, Table 2 shows the neighborhoods $N_i^{\mathbb{C}_j}(x)$ for each x in U and for each combination $(N_i, \mathbb{C}_j)$ with $N_i \in \{N_1, N_2, N_3, N_4\}$ and $\mathbb{C}_j \in \{\mathbb{C}, \mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4, \mathbb{C}_\cap\}$.

We can use the results in Table 2 to establish some differences between neighborhood operators. For example, it can be seen that $N_1^{\mathbb{C}_3} \neq N_4^{\mathbb{C}_\cap}$. In general, we see that there are six groups of possibly equal operators, i.e., based on Example 1, we are not able to distinguish the operators within the same group. These six groups are summarized in Table 3.

We will now consider each of the groups in detail to verify if there are further differences between them. Many results will be directly obtained from Proposition 2.

In group 1, it is very easy to see that $N_1^{\mathbb{C}} = N_1^{\mathbb{C}_1} = N_1^{\mathbb{C}_3} = N_2^{\mathbb{C}_3} = N_1^{\mathbb{C}_\cap}$.

Corollary 1. *Let $\mathbb{C}$ be a covering, then*

(a) $N_1^{\mathbb{C}} = N_1^{\mathbb{C}_1}$,
(b) $N_1^{\mathbb{C}} = N_1^{\mathbb{C}_3} = N_2^{\mathbb{C}_3}$,
(c) $N_1^{\mathbb{C}} = N_1^{\mathbb{C}_\cap}$.

Table 2
Evaluation of 24 neighborhood operators for the covering approximation space $(U, \mathbb{C})$ from Example 1.

Operator	1	2	3	4	Operator	1	2	3	4
$N_1^{\mathbb{C}}$	{1}	{2}	{3}	{4}	$N_1^{\mathbb{C}_1}$	{1}	{2}	{3}	{4}
$N_2^{\mathbb{C}}$	{1}	{1,2,4}	{1,3,4}	{2,3,4}	$N_2^{\mathbb{C}_1}$	{1}	{1,2,4}	{1,3,4}	{2,3,4}
$N_3^{\mathbb{C}}$	{1,2,3}	{2,3}	{2,3}	{2,3,4}	$N_3^{\mathbb{C}_1}$	{1}	{2}	{3}	{4}
$N_4^{\mathbb{C}}$	{1,2,3}	{1,2,3,4}	{1,2,3,4}	{2,3,4}	$N_4^{\mathbb{C}_1}$	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}
$N_1^{\mathbb{C}_2}$	{1,2,3}	{2,3}	{2,3}	{2,3,4}	$N_1^{\mathbb{C}_3}$	{1}	{2}	{3}	{4}
$N_2^{\mathbb{C}_2}$	{1,2,3}	{1,2,3,4}	{1,2,3,4}	{2,3,4}	$N_2^{\mathbb{C}_3}$	{1}	{2}	{3}	{4}
$N_3^{\mathbb{C}_2}$	{1,2,3}	{2,3}	{2,3}	{2,3,4}	$N_3^{\mathbb{C}_3}$	{1}	{2}	{3}	{4}
$N_4^{\mathbb{C}_2}$	{1,2,3}	{1,2,3,4}	{1,2,3,4}	{2,3,4}	$N_4^{\mathbb{C}_3}$	{1}	{2}	{3}	{4}
$N_1^{\mathbb{C}_4}$	{1,2,3}	{2,3}	{2,3}	{2,3,4}	$N_1^{\mathbb{C}_\cap}$	{1}	{2}	{3}	{4}
$N_2^{\mathbb{C}_4}$	{1,2,3}	{1,2,3,4}	{1,2,3,4}	{2,3,4}	$N_2^{\mathbb{C}_\cap}$	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}
$N_3^{\mathbb{C}_4}$	{1,2,3,4}	{1,2,3,4}	{1,2,3,4}	{1,2,3,4}	$N_3^{\mathbb{C}_\cap}$	{1,2,3}	{2,3}	{2,3}	{2,3,4}
$N_4^{\mathbb{C}_4}$	{1,2,3,4}	{1,2,3,4}	{1,2,3,4}	{1,2,3,4}	$N_4^{\mathbb{C}_\cap}$	{1,2,3}	{1,2,3,4}	{1,2,3,4}	{2,3,4}

Table 3
Groups of neighborhood operators based on Example 1.

Group 1	$N_1^{\mathbb{C}}$, $N_1^{\mathbb{C}_1}$, $N_3^{\mathbb{C}_1}$, $N_1^{\mathbb{C}_3}$, $N_2^{\mathbb{C}_3}$, $N_3^{\mathbb{C}_3}$, $N_4^{\mathbb{C}_3}$, $N_1^{\mathbb{C}_\cap}$
Group 2	$N_2^{\mathbb{C}}$, $N_2^{\mathbb{C}_1}$
Group 3	$N_3^{\mathbb{C}}$, $N_1^{\mathbb{C}_2}$, $N_3^{\mathbb{C}_2}$, $N_1^{\mathbb{C}_4}$, $N_3^{\mathbb{C}_\cap}$
Group 4	$N_4^{\mathbb{C}}$, $N_2^{\mathbb{C}_2}$, $N_4^{\mathbb{C}_2}$, $N_2^{\mathbb{C}_4}$, $N_4^{\mathbb{C}_\cap}$
Group 5	$N_3^{\mathbb{C}_4}$, $N_4^{\mathbb{C}_4}$
Group 6	$N_4^{\mathbb{C}_1}$, $N_2^{\mathbb{C}_\cap}$

Proof. These equalities are an immediate result from Propositions 2(a), (c) and (d). □

The last three operators $N_3^{\mathbb{C}_1}$, $N_3^{\mathbb{C}_3}$ and $N_4^{\mathbb{C}_3}$ are not equal to the first five operators from group 1, and they are not equal to each other either, as the following examples show.

Example 2. Let $\mathbb{C} = \{\{1,2\},\{1,2,3\}\}$ be a covering of $U = \{1,2,3\}$, then $N_1^{\mathbb{C}}(1) = \{1,2\}$ and $N_3^{\mathbb{C}_1}(1) = N_3^{\mathbb{C}_3}(1) = N_4^{\mathbb{C}_3}(1) = \{1,2,3\}$. Therefore $N_1^{\mathbb{C}} \neq N_3^{\mathbb{C}_1}$, $N_1^{\mathbb{C}} \neq N_3^{\mathbb{C}_3}$ and $N_1^{\mathbb{C}} \neq N_4^{\mathbb{C}_3}$.

Example 3. Let $\mathbb{C} = \{\{3\},\{1,2\},\{1,3\},\{1,2,3\}\}$ be a covering of $U = \{1,2,3\}$, then $N_3^{\mathbb{C}_1}(1) = \{1\}$ and $N_3^{\mathbb{C}_3}(1) = N_4^{\mathbb{C}_3}(1) = \{1,2\}$. Therefore $N_3^{\mathbb{C}_1} \neq N_3^{\mathbb{C}_3}$ and $N_3^{\mathbb{C}_1} \neq N_4^{\mathbb{C}_3}$.

Example 4. Let $\mathbb{C} = \{\{1\},\{1,2\},\{1,3\}\}$ be a covering of $U = \{1,2,3\}$, then $N_3^{\mathbb{C}_3}(1) = \{1\}$ and $N_4^{\mathbb{C}_3}(1) = \{1,2,3\}$. Therefore $N_3^{\mathbb{C}_3} \neq N_4^{\mathbb{C}_3}$.

We conclude that group 1 consists of four subgroups. The subgroups will be denoted with letters. Hence, group 1 is divided as follows: group *A* consists of neighborhood operators $N_1^{\mathbb{C}}$, $N_1^{\mathbb{C}_1}$, $N_1^{\mathbb{C}_3}$, $N_2^{\mathbb{C}_3}$ and $N_1^{\mathbb{C}_\cap}$, group *B* consists of the operator $N_3^{\mathbb{C}_1}$, group *C* of the operator $N_3^{\mathbb{C}_3}$ and group *D* of the operator $N_4^{\mathbb{C}_3}$.

Next, we show that the two operators in group 2 are equal.

Corollary 2. *Let $\mathbb{C}$ be a covering, then* $N_2^{\mathbb{C}} = N_2^{\mathbb{C}_1}$.

Proof. This follows immediately from Proposition 2(a). □

Thus, group 2 has one subgroup *E* consisting of the operators $N_2^{\mathbb{C}}$ and $N_2^{\mathbb{C}_1}$.

In group 3 are the following four operators equal: $N_3^{\mathbb{C}}, N_1^{\mathbb{C}_2}, N_3^{\mathbb{C}_2}, N_3^{\mathbb{C}_\cap}$.

Corollary 3. *Let $\mathbb{C}$ be a covering, then*

(a) $N_3^{\mathbb{C}} = N_1^{\mathbb{C}_2} = N_3^{\mathbb{C}_2}$,
(b) $N_3^{\mathbb{C}} = N_3^{\mathbb{C}_\cap}$.

Proof. This follows immediately from Propositions 2(b) and (e). □

The fourth operator of this group, $N_1^{\mathbb{C}_4}$, is not equal to the other four.

Example 5. Let $\mathbb{C} = \{\{1,2\},\{2,3\},\{1,3\}\}$, then $N_1^{\mathbb{C}_2}(1) = \{1\}$ and $N_1^{\mathbb{C}_4}(1) = \{1,2,3\}$, so $N_1^{\mathbb{C}_2} \neq N_1^{\mathbb{C}_4}$.

Therefore, the third group consists of two subgroups: group *F* with operators $N_3^{\mathbb{C}}$, $N_1^{\mathbb{C}_2}$, $N_3^{\mathbb{C}_2}$ and $N_3^{\mathbb{C}_\cap}$ and group *G* with operator $N_1^{\mathbb{C}_4}$.

Table 4
Neighborhood operators based on coverings.

Group	Operators	Group	Operators
A.	$N_1^{\mathbb{C}}$, $N_1^{\mathbb{C}_1}$, $N_1^{\mathbb{C}_3}$, $N_2^{\mathbb{C}_3}$, $N_1^{\mathbb{C}_\cap}$	*H.*	$N_4^{\mathbb{C}}$, $N_2^{\mathbb{C}_2}$, $N_4^{\mathbb{C}_2}$, $N_4^{\mathbb{C}_\cap}$
B.	$N_3^{\mathbb{C}_1}$	*I.*	$N_2^{\mathbb{C}_4}$
C.	$N_3^{\mathbb{C}_3}$	*J.*	$N_3^{\mathbb{C}_4}$
D.	$N_4^{\mathbb{C}_3}$	*K.*	$N_4^{\mathbb{C}_4}$
E.	$N_2^{\mathbb{C}}$, $N_2^{\mathbb{C}_1}$	*L.*	$N_4^{\mathbb{C}_1}$
F.	$N_3^{\mathbb{C}}$, $N_1^{\mathbb{C}_2}$, $N_3^{\mathbb{C}_2}$, $N_3^{\mathbb{C}_\cap}$	*M.*	$N_2^{\mathbb{C}_\cap}$
G.	$N_1^{\mathbb{C}_4}$		

For the operators in group 4 it is easy to see that $N_4^{\mathbb{C}} = N_2^{\mathbb{C}_2} = N_4^{\mathbb{C}_2} = N_4^{\mathbb{C}_\cap}$.

Corollary 4. *Let $\mathbb{C}$ be a covering, then*

(a) $N_4^{\mathbb{C}} = N_2^{\mathbb{C}_2} = N_4^{\mathbb{C}_2}$,
(b) $N_4^{\mathbb{C}} = N_4^{\mathbb{C}_\cap}$.

Proof. These equalities are a direct result from Propositions 2(b) and (e). □

The fifth operator is not equal to the other four.

Example 6. Let $\mathbb{C} = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$ be a covering of $U = \{1,2,3,4\}$, then $N_2^{\mathbb{C}_2}(1) = \{1,2,4\}$ and $N_2^{\mathbb{C}_4}(1) = \{1,2,3,4\}$. Therefore $N_2^{\mathbb{C}_2} \neq N_2^{\mathbb{C}_4}$.

The fourth group is therefore decomposed in two subgroups of equal operators: group *H* consists of the operators $N_4^{\mathbb{C}}$, $N_2^{\mathbb{C}_2}$, $N_4^{\mathbb{C}_2}$ and $N_4^{\mathbb{C}_\cap}$ and group *I* consists of operator $N_2^{\mathbb{C}_4}$.

The two operators in group 5 are not equal:

Example 7. Let $\mathbb{C} = \{\{1,2\},\{2,3\},\{1,4\}\}$ be a covering of $U = \{1,2,3,4\}$, then $N_3^{\mathbb{C}_4}(1) = \{1,2\}$ and $N_4^{\mathbb{C}_4}(1) = \{1,2,3,4\}$. Therefore $N_3^{\mathbb{C}_4} \neq N_4^{\mathbb{C}_4}$.

Hence, group 5 consists of two subgroups: group *J* with operator $N_3^{\mathbb{C}_4}$ and group *K* with operator $N_4^{\mathbb{C}_4}$.

Furthermore, the two operators in group 6 are not equal:

Example 8. Let $\mathbb{C} = \{\{1\},\{2,3\},\{1,4\},\{1,2,3\}\}$ be a covering of $U = \{1,2,3,4\}$, then $N_2^{\mathbb{C}_\cap}(1) = \{1,2,3,4\}$ and $N_4^{\mathbb{C}_1}(1) = \{1,4\}$. Therefore $N_2^{\mathbb{C}_\cap} \neq N_4^{\mathbb{C}_1}$.

Hence, group 6 consists of two subgroups: group *L* with operator $N_4^{\mathbb{C}_1}$ and group *M* with operator $N_2^{\mathbb{C}_\cap}$.

Summarizing, we conclude that there are 13 different neighborhood operators which can be obtained by combining the four neighborhood operators and six coverings considered in [27]. We denote them by the letters *A* to *M* and list them in Table 4. Note that the neighborhood operators based on covering $\mathbb{C}_2$ are equal to either $N_3^{\mathbb{C}}$ (group F) or $N_4^{\mathbb{C}}$ (group H), hence, $\mathbb{C}_2$ does not yield new neighborhood operators.

We continue with studying the partial order relation between these 13 groups of neighborhood operators.

3.2. Partial order relation for neighborhood operators

In this section, we establish the Hasse diagram of the 13 groups of neighborhood for the following partial order: let *N* and *N′* be two neighborhood operators, then

$$N \le N' \Leftrightarrow \forall x \in U : N(x) \subseteq N'(x).$$

That is, $N \le N'$ if the neighborhood of any *x* in *U* generated by *N′* contains the one generated by *N* as a subset. This partial order will allow us to distinguish the neighborhood operators that consider relatively larger or smaller collections of associated elements ('neighbors') of a given element.

We start by fixing the type of neighborhood operator and discuss the partial order relation between the operators of type N_i based on different coverings. We begin with the neighborhood operators of type N_1. Recall that the groups of neighborhood operators which contain an operator of type N_1 are *A*, *F* and *G*.

Proposition 3. *Let $\mathbb{C}$ be a covering, then*

(a) $N_1^{\mathbb{C}} \le N_1^{\mathbb{C}_2}$,
(b) $N_1^{\mathbb{C}_2} \le N_1^{\mathbb{C}_4}$.

Proof.

(a) We show that $N_1^{\mathbb{C}}(x) \subseteq N_1^{\mathbb{C}_2}(x)$ for all $x \in U$. By definition, it holds that $\mathbb{C}_2 \subseteq \mathbb{C}$. Furthermore, for $x \in U$ it holds that $\mathcal{C}(\mathbb{C}_2, x) \subseteq \mathcal{C}(\mathbb{C}, x)$. This implies that $\bigcap \mathcal{C}(\mathbb{C}, x) \subseteq \bigcap \mathcal{C}(\mathbb{C}_2, x)$, so $N_1^{\mathbb{C}}(x) \subseteq N_1^{\mathbb{C}_2}(x)$.

(b) Take $x \in U$ and $y \in N_1^{\mathbb{C}_2}(x)$. Then for all $K \in \mathbb{C}_2$ with $x \in K$ it holds that $y \in K$. Take $K' \in \mathcal{C}(\mathbb{C}_4, x)$, then there exist $\{K_i : i \in I\} \subseteq \mathbb{C}_2$ such that $K' = \bigcup_{i \in I} K_i$. Since $x \in K'$, there exists a $i \in I$ such that $x \in K_i$. Hence, $y \in K_i$ and thus $y \in K'$.

We conclude that $y \in N_1^{\mathbb{C}_4}(x)$.

□

Hence, in terms of the notation of Table 4, we conclude that $A \le F \le G$.

We continue with neighborhood operators of type N_2. The groups of neighborhood operators which contain an operator of type N_2 are A, E, H, I and M.

Proposition 4. *Let $\mathbb{C}$ be a covering, then*

(a) $N_2^{\mathbb{C}_3} \le N_2^{\mathbb{C}}$,
(b) $N_2^{\mathbb{C}_3} \le N_2^{\mathbb{C}_\cap}$,
(c) $N_2^{\mathbb{C}} \le N_2^{\mathbb{C}_2}$,
(d) $N_2^{\mathbb{C}_\cap} \le N_2^{\mathbb{C}_2}$,
(e) $N_2^{\mathbb{C}_2} \le N_2^{\mathbb{C}_4}$.

Proof.

(a) Take $x \in U$, then by Proposition 2(c), $N_2^{\mathbb{C}_3}(x) = \bigcap \mathrm{md}(\mathbb{C}, x)$. Hence, $N_2^{\mathbb{C}_3}(x) \subseteq \bigcup \mathrm{md}(\mathbb{C}, x) = N_2^{\mathbb{C}}(x)$.

(b) Take $x \in U$, then by Proposition 2(c), $N_2^{\mathbb{C}_3}(x) = \bigcap \mathrm{md}(\mathbb{C}, x)$ and by Proposition 2(d), $N_2^{\mathbb{C}_3}(x) = \bigcap \mathrm{md}(\mathbb{C}_\cap, x)$. Hence, $N_2^{\mathbb{C}_3}(x) \subseteq \bigcup \mathrm{md}(\mathbb{C}_\cap, x) = N_2^{\mathbb{C}_\cap}(x)$.

(c) Take $x \in U$ and $y \in N_2^{\mathbb{C}}(x)$, then there exists a $K \in \mathrm{md}(\mathbb{C}, x)$ with $y \in K$. Hence, there exists a $K' \in \mathrm{MD}(\mathbb{C}, x)$ with $K \subseteq K'$ and thus, $y \in K'$. By Proposition 2(b), $\mathrm{MD}(\mathbb{C}, x) = \mathrm{md}(\mathbb{C}_2, x)$. Thus, $K' \in \mathrm{md}(\mathbb{C}_2, x)$ and $y \in K'$. Hence, $y \in N_2^{\mathbb{C}_2}(x)$.

(d) Take $x \in U$ and $y \in N_2^{\mathbb{C}_\cap}(x)$, then there exists a $K \in \mathrm{md}(\mathbb{C}_\cap, x)$ with $y \in K$. Since $K \in \mathbb{C}_\cap \subseteq \mathbb{C}$, there exists a $K' \in \mathrm{MD}(\mathbb{C}, x)$ with $K \subseteq K'$ and thus, $y \in K'$. By Proposition 2(b), $\mathrm{MD}(\mathbb{C}, x) = \mathrm{md}(\mathbb{C}_2, x)$. Thus, $K' \in \mathrm{md}(\mathbb{C}_2, x)$ and $y \in K'$. Hence, $y \in N_2^{\mathbb{C}_2}(x)$.

(e) By Corollary 4(a) it holds that $N_2^{\mathbb{C}_2} = N_4^{\mathbb{C}}$. Moreover, we can write $\mathbb{C}_4 = \{N_4^{\mathbb{C}}(x) : x \in U\}$.

Take $x \in U$, we will prove that $N_4^{\mathbb{C}}(x) \subseteq N_2^{\mathbb{C}_4}(x)$.

Denote $\mathrm{md}(\mathbb{C}_4, x) = \{N_4^{\mathbb{C}}(z_i) : z_i \in U, i \in I\}$ and assume there is an element $y \in U$ with $y \in N_4^{\mathbb{C}}(x) \setminus N_2^{\mathbb{C}_4}(x)$, i.e., $y \in N_4^{\mathbb{C}}(x)$ and for all $i \in I : y \notin N_4^{\mathbb{C}}(z_i)$. Hence, there exists a $K^* \in \mathbb{C}$ with $x, y \in K^*$ and for all $i \in I$ and for all $K \in \mathbb{C}$ it holds that if $y \in K$, then $z_i \notin K$. In other words, $x \in N_4^{\mathbb{C}}(y)$ and $z_i \notin N_4^{\mathbb{C}}(y)$ for all $i \in I$. Since $N_4^{\mathbb{C}}(y) \in \mathbb{C}_4$ and $x \in N_4^{\mathbb{C}}(y)$, there exists a $L \in \mathrm{md}(\mathbb{C}_4, x)$ with $L \subseteq N_4^{\mathbb{C}}(y)$. Since $L \in \mathrm{md}(\mathbb{C}_4, x)$, $L = N_4^{\mathbb{C}}(z_i)$ for some $i \in I$, and thus, $N_4^{\mathbb{C}}(z_i) \subseteq N_4^{\mathbb{C}}(y)$. Hence, $z_i \in N_4^{\mathbb{C}}(y)$, which is a contradiction. We conclude that $N_4^{\mathbb{C}}(x) \setminus N_2^{\mathbb{C}_4}(x) = \emptyset$ and thus, $N_4^{\mathbb{C}}(x) \subseteq N_2^{\mathbb{C}_4}(x)$.

□

However, $N_2^{\mathbb{C}}$ and $N_2^{\mathbb{C}_\cap}$ are incomparable.

Example 9. If $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1\}, \{1, 2\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\}$, then $N_2^{\mathbb{C}}(2) = \{1, 2, 3, 4\}$ and $N_2^{\mathbb{C}_\cap}(2) = \{2, 3, 4\}$. Furthermore, if $\mathbb{C} = \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}\}$, then $N_2^{\mathbb{C}}(1) = \{1\}$ and $N_2^{\mathbb{C}_\cap}(1) = \{1, 2, 3\}$.

We can conclude that $A \le E \le H \le I$ and $A \le M \le H \le I$.

Next, we discuss neighborhood operators of type N_3 by considering groups B, C, F and J.

Proposition 5. *Let $\mathbb{C}$ be a covering, then $N_3^{\mathbb{C}} \le N_3^{\mathbb{C}_4}$.*

Proof. Take $x \in U$ and $y \in N_3^{\mathbb{C}}(x)$, then for all $K \in \mathrm{MD}(\mathbb{C}, x)$ it holds that $y \in K$. Take $L \in \mathrm{MD}(\mathbb{C}_4, x)$. Since $L \in \mathbb{C}_4$, there exists a $z \in U$ such that $L = \bigcup \mathrm{MD}(\mathbb{C}, z)$. Since $x \in L$, there exists a $L' \in \mathrm{MD}(\mathbb{C}, z)$ such that $x \in L'$ and thus, there exists a $L'' \in \mathrm{MD}(\mathbb{C}, x)$ such that $L' \subseteq L''$. Since $L' \in \mathrm{MD}(\mathbb{C}, z)$, we have that $L' = L''$, thus $L' \in \mathrm{MD}(\mathbb{C}, x)$ and therefore $y \in L'$. We conclude that $y \in L$ and $y \in N_3^{\mathbb{C}_4}(x)$. □

On the other hand, the neighborhood operator $N_3^{\mathbb{C}_3}$ is incomparable with the neighborhood operators $N_3^{\mathbb{C}}$ and $N_3^{\mathbb{C}_1}$.

Example 10. If $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1\}, \{2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 3, 4\}\}$, then $N_3^{\mathbb{C}}(4) = N_3^{\mathbb{C}_1}(4) = \{4\}$ and $N_3^{\mathbb{C}_3}(4) = \{3, 4\}$. Furthermore, if $\mathbb{C} = \{\{1\}, \{4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3\}\}$, then $N_3^{\mathbb{C}}(4) = N_3^{\mathbb{C}_1}(4) = \{1, 4\}$ and $N_3^{\mathbb{C}_3}(4) = \{4\}$.

Moreover, $N_3^{\mathbb{C}}$ and $N_3^{\mathbb{C}_1}$ are also incomparable.

Example 11. If $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}\}$, then $N_3^{\mathbb{C}}(1) = \{1, 2, 3\}$ and $N_3^{\mathbb{C}_1}(1) = \{1\}$. Furthermore, if $\mathbb{C} = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\}$, then $N_3^{\mathbb{C}}(1) = \{1, 2\}$ and $N_3^{\mathbb{C}_1}(1) = \{1, 2, 4\}$.

Furthermore, $N_3^{\mathbb{C}_1}$ and $N_3^{\mathbb{C}_4}$ are likewise incomparable.

Example 12. If $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 4\}\}$, then $N_3^{\mathbb{C}_1}(1) = \{1, 4\}$ and $N_3^{\mathbb{C}_4}(1) = \{1, 2\}$.

To end the analysis of groups *B*, *C*, *F* and *J*, we show that $N_3^{\mathbb{C}_3}$ and $N_3^{\mathbb{C}_4}$ are also incomparable.

Example 13. If $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1, 3\}, \{2, 3\}, \{1, 4\}\}$, then $N_3^{\mathbb{C}_3}(1) = \{1, 4\}$ and $N_3^{\mathbb{C}_4}(1) = \{1, 3\}$.

Hence, we conclude that the only partial order relation between groups *B*, *C*, *F* and *J* is $F \leq J$.

Finally, we discuss the order relation for neighborhood operators of type N_4. The groups of neighborhood operators which contain an operator of type N_4 are *D*, *H*, *K* and *L*.

Proposition 6. *Let $\mathbb{C}$ be a covering, then*

(a) $N_4^{\mathbb{C}_3} \leq N_4^{\mathbb{C}_1}$,
(b) $N_4^{\mathbb{C}_1} \leq N_4^{\mathbb{C}}$,
(c) $N_4^{\mathbb{C}} \leq N_4^{\mathbb{C}_4}$.

Proof.

(a) Take $x \in U$ and $y \in N_4^{\mathbb{C}_3}(x)$, then there exists a $K \in \mathrm{MD}(\mathbb{C}_3, x)$ with $y \in K$. Then there exists a $K' \in \mathbb{C}_1$ with $K \subseteq K'$ and there exists a $K'' \in \mathrm{MD}(\mathbb{C}_1, x)$ with $K' \subseteq K''$. Thus, $y \in K'' \subseteq N_4^{\mathbb{C}_1}(x)$.
(b) Take $x \in U$ and $y \in N_4^{\mathbb{C}_1}(x)$, then there exists a $K \in \mathrm{MD}(\mathbb{C}_1, x)$ with $y \in K$. Thus, $K \in \mathbb{C}_1 \subseteq \mathbb{C}$, so there exists a $K' \in \mathrm{MD}(\mathbb{C}, x)$ with $K \subseteq K'$. Hence, $y \in K'$ and $y \in N_4^{\mathbb{C}}(x)$.
(c) Take $x \in U$ and $y \in N_4^{\mathbb{C}}(x)$, then there exists a $K \in \mathrm{MD}(\mathbb{C}, x)$ with $y \in K$. Take $K' = \bigcup \mathrm{MD}(\mathbb{C}, x) \in \mathbb{C}_4$, then $y \in K'$ and there exists a $K'' \in \mathrm{MD}(\mathbb{C}_4, x)$ with $K' \subseteq K''$. Hence, $y \in K'' \subseteq N_4^{\mathbb{C}_4}(x)$.

□

In terms of Table 4, this means that $D \leq L \leq H \leq K$.

Besides fixing the choice of type of neighborhood operator, it is also possible to fix the covering. For a fixed covering $\mathbb{C}$, the following order relations for neighborhood operators have been established in [10]:

Proposition 7. *[10] Let $\mathbb{C}$ be a covering, it holds that*

(a) $N_1^{\mathbb{C}} \leq N_2^{\mathbb{C}} \leq N_4^{\mathbb{C}}$,
(b) $N_1^{\mathbb{C}} \leq N_3^{\mathbb{C}} \leq N_4^{\mathbb{C}}$.

These inequalities show that the operators N_1 and N_4 result in the smallest and largest neighborhoods, given a covering $\mathbb{C}$.

Corollary 5. *Let $\mathbb{C}$ be a covering, then for $\mathbb{C}_1$ we obtain that*

- $N_2^{\mathbb{C}_1} \leq N_4^{\mathbb{C}_1}$,
- $N_1^{\mathbb{C}_1} \leq N_3^{\mathbb{C}_1} \leq N_4^{\mathbb{C}_1}$.

Moreover, for $\mathbb{C}_3$ it holds that

- $N_1^{\mathbb{C}_3} \leq N_3^{\mathbb{C}_3} \leq N_4^{\mathbb{C}_3}$,

and for $\mathbb{C}_4$ that

- $N_2^{\mathbb{C}_4} \leq N_4^{\mathbb{C}_4}$,
- $N_1^{\mathbb{C}_4} \leq N_3^{\mathbb{C}_4} \leq N_4^{\mathbb{C}_4}$.

In terms of Table 4, we have that $E \leq L$, $A \leq B \leq L$, $A \leq C \leq D$, $I \leq K$ and $G \leq J \leq K$.

Until now, we either fixed the covering, or the type of neighborhood operator. However, we also have the following order relation between $N_1^{\mathbb{C}_4}$ and $N_4^{\mathbb{C}}$:

Proposition 8. *Let $\mathbb{C}$ be a covering, then $N_1^{\mathbb{C}_4} \leq N_4^{\mathbb{C}}$.*

Proof. For $x \in U$, $N_4^{\mathbb{C}}(x) \in \mathbb{C}_4$ and $x \in N_4^{\mathbb{C}}(x)$. Thus, there exists a $L \in \mathrm{md}(\mathbb{C}_4, x)$ with $L \subseteq N_4^{\mathbb{C}}(x)$. Hence, $N_1^{\mathbb{C}_4}(x) \subseteq L \subseteq N_4^{\mathbb{C}}(x)$. □

We conclude that $G \leq H$.

Table 5
Incomparable neighborhood operators.

$(U, \mathbb{C})_a$ and $(U, \mathbb{C})_b$	Elements	Incomparable pairs
$(U, \mathbb{C})_1$ and $(U, \mathbb{C})_1$	$x = 1$	(*D*, *J*), (*J*, *L*)
$(U, \mathbb{C})_1$ and $(U, \mathbb{C})_1$	$x = 1, 2$	(*B*, *M*), (*C*, *M*), (*D*, *M*), (*J*, *M*), (*L*, *M*)
$(U, \mathbb{C})_1$ and $(U, \mathbb{C})_1$	$x = 1, 3$	(*H*, *J*), (*I*, *J*)
$(U, \mathbb{C})_1$ and $(U, \mathbb{C})_2$	$x = 1$	(*B*, *G*), (*C*, *G*), (*D*, *F*), (*D*, *G*)
$(U, \mathbb{C})_1$ and $(U, \mathbb{C})_3$	$x = 1$	(*B*, *E*), (*C*, *E*), (*D*, *E*), (*E*, *J*)
$(U, \mathbb{C})_4$ and $(U, \mathbb{C})_4$	$x = 1, 3$	(*B*, *D*)

There are no other order relations between different groups of neighborhood operators than those established above. In order to see this, we obtain from Example 1 that the pairs (*E*, *F*), (*E*, *G*), (*F*, *L*), (*F*, *M*), (*G*, *L*) and (*G*, *M*) are incomparable. From Example 9, we derive that (*E*, *M*) is incomparable and from Example 10, it is clear that (*B*, *C*) and (*C*, *F*) are incomparable pairs. Furthermore, (*B*, *F*), (*B*, *J*) and (*C*, *J*) are incomparable by Examples 11, 12 and 13, respectively. Finally, the incomparability of the remaining pairs can be derived from Example 14.

Example 14. Given the following covering-based approximation spaces:

- $(U, \mathbb{C})_1$ with $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 4\}\}$,
- $(U, \mathbb{C})_2$ with $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}\}$,
- $(U, \mathbb{C})_3$ with $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1, 2\}, \{2, 3\}, \{1, 4\}\}$,
- $(U, \mathbb{C})_4$ with $U = \{1, 2, 3\}$ and $\mathbb{C} = \{\{3\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$,

In Table 5, we present a compact way to illustrate that the 18 remaining pairs of neighborhood operators are all incomparable. For example, to show that *D* and *J* are incomparable, we consider $(U, \mathbb{C})_1$ with $x = 1$. As $D(1) = \{1, 4\}$ and $J(1) = \{1, 2\}$, we see that *D* and *J* are not comparable with each other. On the other hand, to study the comparability between *B* and *G*, we see that in $(U, \mathbb{C})_1$ we have $B(1) = \{1, 4\}$ and $G(1) = \{1\}$, while in $(U, \mathbb{C})_2$ it holds that $B(1) = \{1\}$ and $G(1) = \{1, 2, 3\}$, hence, *B* and *G* are incomparable. The incomparability of the 16 other pairs can be studied in a similar way.

In Fig. 2, the resulting Hasse diagram presenting the partial order between the 13 groups of neighborhood operators is shown. Note that this Hasse diagram is in fact a lattice, i.e., for every two neighborhood operators, there is an neighborhood operator larger than, as well as an operator smaller than those two operators. This can be easily seen by the fact that the Hasse diagram yields a minimum and a maximum neighborhood operator. The former is $N_1^{\mathbb{C}}$ (group *A*). This is in line with our intuition, if we look at its definition: let $x \in U$, then

$$N_1^{\mathbb{C}}(x) = \bigcap \{K \in \mathbb{C} : x \in K\}.$$

That is, $N_1^{\mathbb{C}}(x)$ contains only those elements which appear in *every* *K* of the covering $\mathbb{C}$ that contains *x*, in other words, elements which are always associated to *x*. On the other hand, the largest neighborhoods are generated by $N_4^{\mathbb{C}_4}$ (group *K*)

$$N_4^{\mathbb{C}_4}(x) = \bigcup \{K \in \mathbb{C}_4 : x \in K\}.$$

This means, for *y* in *U*,

$$\begin{aligned} y \in N_4^{\mathbb{C}_4}(x) &\iff \exists K \in \mathbb{C}_4 : x, y \in K \\ &\iff \exists z \in U : K = \bigcup \mathcal{C}(\mathbb{C}, z) \wedge x, y \in K \\ &\iff \exists z \in U, K_1, K_2 \in \mathcal{C}(\mathbb{C}, z) : x \in K_1 \wedge y \in K_2 \\ &\iff \exists z \in U, K_1 \in \mathcal{C}(\mathbb{C}, x), K_2 \in \mathcal{C}(\mathbb{C}, y) : z \in K_1 \cap K_2 \end{aligned}$$

In other words, an element *y* in *U* belongs to $N_4^{\mathbb{C}_4}(x)$ if there exists some *z* in *U* that appears together with *x* in some element K_1 of the covering $\mathbb{C}$ and with *y* in some element K_2 (not necessarily equal to K_1) of the covering $\mathbb{C}$. A real-life example of the neighborhood operator $N_4^{\mathbb{C}_4}$ is the following: let *U* be the students in one grade class and let the sets of the covering $\mathbb{C}$ represent the classes in a certain school year over a period of five years. Then $y \in N_4^{\mathbb{C}_4}(x)$ if *x* and *z* were in a class some year and *y* and *z* were in a class some year, but *x* and *y* were not necessarily classmates. Note how the neighborhood $N_4^{\mathbb{C}_4}(x)$ is different from $N_4^{\mathbb{C}}(x)$, to which an element *y* in *U* belongs if it coincides with *x* in at least one of the sets of the covering $\mathbb{C}$, a stricter condition (*x* and *y* had to be classmates in a certain school year).

In the next section, we discuss how we can apply the lattice in Fig. 2 to covering-based rough set approximations.

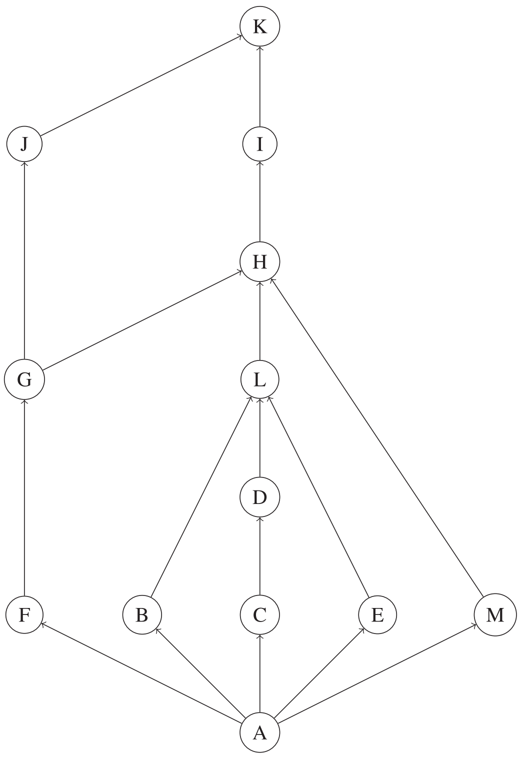

Fig. 2. Partial order of neighborhood operators.

4. Application to covering-based rough set approximation operators

In this section, we first study which of the 13 groups of neighborhood operators lead to new approximation operators. Furthermore, we extend Table 1. In addition, we study the partial order relation of all covering-based rough set approximation operators by extending Fig. 1 with these new operators.

4.1. New covering-based rough set approximation operators

The 13 different groups of neighborhood operators based on coverings result in 13 dual pairs of element-based rough sets $(\underline{\mathrm{apr}}^{\mathbb{C}_j}_{N_i}, \overline{\mathrm{apr}}^{\mathbb{C}_j}_{N_i})$ according to Eqs. (3) and (4). There is a one-to-one correspondence between different neighborhood operators and different element-based approximation operators:

Proposition 9. *Let N and N' be two neighborhood operators, then $N = N' \Leftrightarrow \overline{\mathrm{apr}}_N = \overline{\mathrm{apr}}_{N'}$.*

Proof. It is easy to see that if $N = N'$, then $\overline{\mathrm{apr}}_N = \overline{\mathrm{apr}}_{N'}$. Now assume that $\overline{\mathrm{apr}}_N = \overline{\mathrm{apr}}_{N'}$, we will prove that $N(x) = N'(x)$ for $x \in U$. Let $y \in N(x)$, then $N(x) \cap \{y\} \neq \emptyset$. Hence, $x \in \overline{\mathrm{apr}}_N(\{y\})$ and thus, $x \in \overline{\mathrm{apr}}_{N'}(\{y\})$. So, $N'(x) \cap \{y\} \neq \emptyset$ and thus, $y \in N'(x)$. In similar way we can prove that $N'(x) \subseteq N(x)$. □

As mentioned in Section 2, in [10], Restrepo et al. identified 16 different pairs of covering-based rough set approximation operators. From Proposition 9, we obtain that the approximation operators of pairs 1, 2, 3 and 4 from Table 1 correspond to the neighborhood operators of the groups A, E, F and H, respectively. Moreover, in the same way that $\underline{\mathrm{apr}}''_{\mathbb{C}}$ (pair 4) is related to operator $N_4^{\mathbb{C}}$ (group H), the pairs 9, 10 and 11 are related to the neighborhood operators of the groups L, D and K, respectively.

The six remaining groups of neighborhood operators based on coverings, namely groups B, C, G, I, J and M, result in new approximation operators. This can be seen as follows: all lower approximation operators based on neighborhoods are complete meet morphisms [10], i.e., let N be a neighborhood operator and $A_i \subseteq U$ for $i \in I$, then

$$\underline{\mathrm{apr}}_N\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} \underline{\mathrm{apr}}_N(A_i).$$

Furthermore, the lower approximation operators of pairs 5–8, 12, 13 and 15 are no meet morphisms, while 14 and 16 are [10]. Hence, this observation only leaves the possible equality of some of the new approximation operators to pair 14 or

Table 6
Overview of covering-based rough set approximation operators.

Number	Group	Lower approximation	Upper approximation
1	A	$\underline{\mathrm{apr}}^{\mathbb{C}}_{N_1} = \underline{\mathrm{apr}}'_{\mathbb{C}_3} = \underline{\mathrm{apr}}^{\mathbb{C}_1}_{N_1} = \underline{\mathrm{apr}}^{\mathbb{C}_3}_{N_1} = \underline{\mathrm{apr}}^{\mathbb{C}_3}_{N_2} = \underline{\mathrm{apr}}^{\mathbb{C}_\cap}_{N_1}$	$\overline{\mathrm{apr}}^{\mathbb{C}}_{N_1} = \overline{\mathrm{apr}}'_{\mathbb{C}_3} = \overline{\mathrm{apr}}^{\mathbb{C}_1}_{N_1} = \overline{\mathrm{apr}}^{\mathbb{C}_3}_{N_1} = \overline{\mathrm{apr}}^{\mathbb{C}_3}_{N_2} = \overline{\mathrm{apr}}^{\mathbb{C}_\cap}_{N_1}$
2	E	$\underline{\mathrm{apr}}^{\mathbb{C}}_{N_2} = \underline{\mathrm{apr}}^{\mathbb{C}_1}_{N_2}$	$\overline{\mathrm{apr}}^{\mathbb{C}}_{N_2} = \overline{\mathrm{apr}}^{\mathbb{C}_1}_{N_2}$
3	F	$\underline{\mathrm{apr}}^{\mathbb{C}}_{N_3} = \underline{\mathrm{apr}}^{\mathbb{C}_2}_{N_1} = \underline{\mathrm{apr}}^{\mathbb{C}_2}_{N_3} = \underline{\mathrm{apr}}^{\mathbb{C}_\cap}_{N_3}$	$\overline{\mathrm{apr}}^{\mathbb{C}}_{N_3} = \overline{\mathrm{apr}}^{\mathbb{C}_2}_{N_1} = \overline{\mathrm{apr}}^{\mathbb{C}_2}_{N_3} = \overline{\mathrm{apr}}^{\mathbb{C}_\cap}_{N_3}$
4	H	$\underline{\mathrm{apr}}^{\mathbb{C}}_{N_4} = \underline{\mathrm{apr}}''_{\mathbb{C}} = \underline{\mathrm{apr}}''_{\mathbb{C}_2} = \underline{\mathrm{apr}}''_{\mathbb{C}_\cap} = \underline{\mathrm{apr}}^{\mathbb{C}_2}_{N_2} = \underline{\mathrm{apr}}^{\mathbb{C}_2}_{N_4} = \underline{\mathrm{apr}}^{\mathbb{C}_\cap}_{N_4}$	$\overline{\mathrm{apr}}^{\mathbb{C}}_{N_4} = \overline{\mathrm{apr}}''_{\mathbb{C}} = \overline{\mathrm{apr}}''_{\mathbb{C}_2} = \overline{\mathrm{apr}}''_{\mathbb{C}_\cap} = \overline{\mathrm{apr}}^{\mathbb{C}_2}_{N_2} = \overline{\mathrm{apr}}^{\mathbb{C}_2}_{N_4} = \overline{\mathrm{apr}}^{\mathbb{C}_\cap}_{N_4}$
5		$\underline{\mathrm{apr}}'_{\mathbb{C}} = \underline{\mathrm{apr}}'_{\mathbb{C}_1} = \underline{\mathrm{apr}}'_{S_\cup}$	$\overline{\mathrm{apr}}'_{\mathbb{C}} = \overline{\mathrm{apr}}'_{\mathbb{C}_1} = \overline{\mathrm{apr}}'_{S_\cup}$
6		$\underline{\mathrm{apr}}'_{\mathbb{C}_2}$	$\overline{\mathrm{apr}}'_{\mathbb{C}_2}$
7		$\underline{\mathrm{apr}}'_{\mathbb{C}_4}$	$\overline{\mathrm{apr}}'_{\mathbb{C}_4}$
8		$\underline{\mathrm{apr}}'_{\mathbb{C}_\cap}$	$\overline{\mathrm{apr}}'_{\mathbb{C}_\cap}$
9	L	$\underline{\mathrm{apr}}''_{\mathbb{C}_1} = \underline{\mathrm{apr}}^{\mathbb{C}_1}_{N_4}$	$\overline{\mathrm{apr}}''_{\mathbb{C}_1} = \overline{\mathrm{apr}}^{\mathbb{C}_1}_{N_4}$
10	D	$\underline{\mathrm{apr}}''_{\mathbb{C}_3} = \underline{\mathrm{apr}}^{\mathbb{C}_3}_{N_4}$	$\overline{\mathrm{apr}}''_{\mathbb{C}_3} = \overline{\mathrm{apr}}^{\mathbb{C}_3}_{N_4}$
11	K	$\underline{\mathrm{apr}}''_{\mathbb{C}_4} = \underline{\mathrm{apr}}^{\mathbb{C}_4}_{N_4}$	$\overline{\mathrm{apr}}''_{\mathbb{C}_4} = \overline{\mathrm{apr}}^{\mathbb{C}_4}_{N_4}$
12		$\underline{\mathrm{apr}}_{S_\cap}$	$\overline{\mathrm{apr}}_{S_\cap}$
13		$(H_1^{\mathbb{C}})^\partial$	$H_1^{\mathbb{C}}$
14		$(H_3^{\mathbb{C}})^\partial$	$H_3^{\mathbb{C}}$
15		$(H_4^{\mathbb{C}})^\partial$	$H_4^{\mathbb{C}}$
16		$(H_5^{\mathbb{C}})^\partial$	$H_5^{\mathbb{C}}$
17	B	$\underline{\mathrm{apr}}^{\mathbb{C}_1}_{N_3}$	$\overline{\mathrm{apr}}^{\mathbb{C}_1}_{N_3}$
18	C	$\underline{\mathrm{apr}}^{\mathbb{C}_3}_{N_3}$	$\overline{\mathrm{apr}}^{\mathbb{C}_3}_{N_3}$
19	G	$\underline{\mathrm{apr}}^{\mathbb{C}_4}_{N_1}$	$\overline{\mathrm{apr}}^{\mathbb{C}_4}_{N_1}$
20	I	$\underline{\mathrm{apr}}^{\mathbb{C}_4}_{N_2}$	$\overline{\mathrm{apr}}^{\mathbb{C}_4}_{N_2}$
21	J	$\underline{\mathrm{apr}}^{\mathbb{C}_4}_{N_3}$	$\overline{\mathrm{apr}}^{\mathbb{C}_4}_{N_3}$
22	M	$\underline{\mathrm{apr}}^{\mathbb{C}_\cap}_{N_2}$	$\overline{\mathrm{apr}}^{\mathbb{C}_\cap}_{N_2}$

pair 16. It can be seen that is not the case for group B: from the results obtained in Section 3.2, it holds for all x in U that $N_1^{\mathbb{C}}(x) \subseteq N_3^{\mathbb{C}_1}(x)$. Therefore, by Eq. (4) we derive that

$$\forall A \subseteq U : \overline{\mathrm{apr}}^{\mathbb{C}}_{N_1}(A) \subseteq \overline{\mathrm{apr}}^{\mathbb{C}_1}_{N_3}(A).$$

This means that the pair of approximation operators obtained from the neighborhood operator $N_3^{\mathbb{C}_1}$ is comparable with the approximation operators from pair 1. However, as can be seen in Fig. 1, neither pair 14 nor pair 16 is comparable with pair 1. A similar reasoning holds for groups C, G, I, J and M.

In Table 6, we extend Table 1 by adding the element-based rough sets obtained by the 13 groups of neighborhood operators.

4.2. Hasse diagram of covering-based rough set approximation operators

Next, we will add the six new covering-based approximation operators to the Hasse diagram from Fig. 1 showing the partial order $\leq$ of lower approximation operators, defined in Eq. (18). Note that since all pairs are dual, it suffices to consider the lower approximation operators. The results for the upper approximation operators are analogous.

The study of the Hasse diagram of the lower approximation operators gives an indication on the accuracy of the approximation operators. A measure for accuracy is mostly given by the ratio between the cardinality of the lower and the upper approximation [7]: given $A \subseteq U$, the accuracy of the dual pair of approximation operators $(\underline{\mathrm{apr}}, \overline{\mathrm{apr}})$ for A is given by $\frac{|\underline{\mathrm{apr}}(A)|}{|\overline{\mathrm{apr}}(A)|}$. Thus, the pair $(\underline{\mathrm{apr}}_1, \overline{\mathrm{apr}}_1)$ is more accurate than the pair $(\underline{\mathrm{apr}}_2, \overline{\mathrm{apr}}_2)$ if

$$\forall A \subseteq U : \frac{|\underline{\mathrm{apr}}_2(A)|}{|\overline{\mathrm{apr}}_2(A)|} \leq \frac{|\underline{\mathrm{apr}}_1(A)|}{|\overline{\mathrm{apr}}_1(A)|}.$$

By the fact that

$$\forall A \subseteq U : \overline{\mathrm{apr}}_1(A) \subseteq \overline{\mathrm{apr}}_2(A) \Leftrightarrow \underline{\mathrm{apr}}_2(A) \subseteq \underline{\mathrm{apr}}_1(A),$$

it is sufficient to study the order relation of the lower approximation operators: the larger the lower approximation operator, the more accurate the dual operators.

In order to establish the order relation of the 22 pairs, we have to study the comparability between pairs 17–22 and pairs 5–8 and 12–16, since the comparability between pairs 17–22 and pairs 1–4 and 9–11 can be derived from Fig. 2 and Proposition 10 below. Pair X and pair Y are comparable with each other if $X \leq Y$ or $Y \leq X$. For the sake of notation, we will only write the number of the pair of the corresponding lower approximation operator.

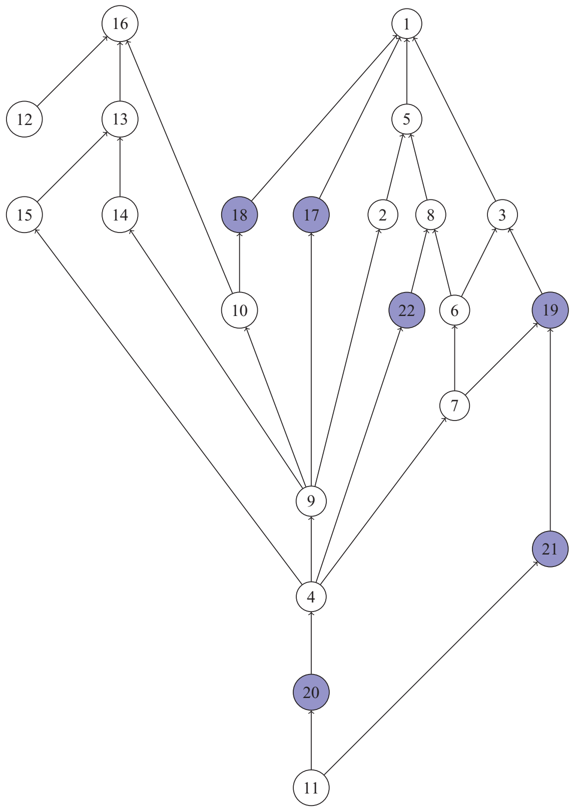

Fig. 3. Partial ordering for lower approximations operators from Table 6.

Proposition 10. *[10] Let N and M be two neighborhood operators such that* $N \subseteq M$, *then* $\underline{\text{apr}}_M \leq \underline{\text{apr}}_N$.

Corollary 6. *From Proposition 10, we derive the following results:*

(a) $9 \preceq 17 \preceq 1$,
(b) $10 \preceq 18 \preceq 1$,
(c) $4 \preceq 19 \preceq 3$,
(d) $11 \preceq 20 \preceq 4$,
(e) $11 \preceq 21 \preceq 19$,
(f) $4 \preceq 22 \preceq 1$.

Corollary 6 gives an indication where to put the six new approximation operators in the Hasse diagram. Based on Corollary 6(d), we conclude that pair 20 is comparable with pairs 5–8 and 13–16, since it is comparable with pair 4. Furthermore, based on Fig. 1 and Corollary 6, we can exclude several partial order relations at once. For example, for the pairs 20 and 12, it cannot hold that $20 \preceq 12$, otherwise $11 \preceq 12$ would also hold, which conflicts with the results from Fig. 1. Moreover, $12 \preceq 20$ can also not hold, otherwise $12 \preceq 1$. Therefore, the pairs 12 and 20 are incomparable and we can add pair 20 to Fig. 3.

Furthermore, we have the following two propositions:

Proposition 11. *Let* $\mathbb{C}$ *be a covering, then* $\underline{\text{apr}}'_{\mathbb{C}_4} \leq \underline{\text{apr}}^{\mathbb{C}_4}_{N_1}$.

Proof. Let $A \subseteq U$ and $x \in \underline{\text{apr}}'_{\mathbb{C}_4}(A)$, then there exists a $K \in \mathbb{C}_4$ with $K \subseteq A$ and $x \in K$. Since $N_1^{\mathbb{C}_4}(x) \subseteq K$ it holds that $N_1^{\mathbb{C}_4}(x) \subseteq A$. Hence, $x \in \underline{\text{apr}}^{\mathbb{C}_4}_{N_1}(A)$. □

Thus, pair 19 is comparable with pair 7: $7 \preceq 19$. For pairs 8 and 22, we obtain the following:

Proposition 12. *Let* $\mathbb{C}$ *be a covering, then* $\underline{\text{apr}}^{\mathbb{C}_\cap}_{N_2} \leq \underline{\text{apr}}'_{\mathbb{C}_\cap}$.

Proof. Let $A \subseteq U$ and $x \in U$ such that $x \in \underline{\text{apr}}^{\mathbb{C}_\cap}_{N_2}(A)$. Then $N_2^{\mathbb{C}_\cap}(x) \subseteq A$ and thus, for all $K \in \text{md}(\mathbb{C}_\cap, x)$ it holds that $K \subseteq A$. Hence, there exists a $K \in \mathbb{C}_\cap$ with $x \in K$ and $K \subseteq A$, thus $x \in \underline{\text{apr}}'_{\mathbb{C}_\cap}(A)$. □

Hence, pair 22 is comparable with pair 8: 22 ≤ 8. From Fig. 1, we derive as well that 22 ≤ 5, thus, pair 22 is comparable with pair 5.

There are no other comparability statements between the pairs 17–22 and the pairs 5–8 and 12–16. Counterexamples can be found in Appendix A. Hence, the Hasse diagram of the 22 pairs from Table 6 is established in Fig. 3. Taking into account the accuracy, we see that the pairs 1 and 16 result in the most accurate approximation operators, while pair 11 yields the least accuracy.

In the next section, we study the connection between covering- and relation-based rough set approximation operators.

5. Connection with relation-based rough set approximation operators

An alternative generalization of Pawlak's rough sets [7] is obtained by replacing the equivalence relation E with an arbitrary binary relation R in U. Below, we recall the definition of the corresponding rough set approximation operators:

Definition 8. [25] Let R be a binary relation on U and $A \subseteq U$, then the lower and upper approximation of A based on R is given by

$$\underline{\mathrm{apr}}_R(A) = \{x \in U : R(x) \subseteq A\}, \tag{19}$$

$$\overline{\mathrm{apr}}_R(A) = \{x \in U : R(x) \cap A \neq \emptyset\}, \tag{20}$$

with $R(x) = \{y \in U : xRy\}$. The pair (U, R) is called a relation approximation space.

Note that each relation-based rough set model is an element-based rough set model as well, by taking the neighborhood $N(x) = R(x)$.

The purpose of this section is twofold: first, we want to verify which of the 22 pairs of covering-based rough set operators from Table 6 can be connected to relation-based approximation operators (Section 5.1). Next, for those pairs of operators that can also be generated by a binary relation, we want to study which properties (reflexive, symmetric, transitive) the corresponding relation has (Section 5.2). As we will see, the answers to these questions are tightly linked to the existence and properties of neighborhood operators generating a specific pair of approximation operators.

5.1. Generating relations with neighborhood operators

In order to study the connection with relation-based approximation operators, we recall the proposition of Yao [24] who proved a sufficient and necessary condition for the equivalence between general approximation operators and relation-based ones.

Proposition 13. *[24] Suppose* $(\underline{\mathrm{apr}}, \overline{\mathrm{apr}})$ *is a dual pair of approximation operators on U. The upper approximation operator* $\overline{\mathrm{apr}}$ *satisfies* $\overline{\mathrm{apr}}(\emptyset) = \emptyset$ *and for all* $A_i \subseteq U$, $i \in I$, : $\overline{\mathrm{apr}}\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \overline{\mathrm{apr}}(A_i)$, *i.e.,* $\overline{\mathrm{apr}}$ *is a complete join morphism, if and only if there exists a relation R such that* $(\underline{\mathrm{apr}}, \overline{\mathrm{apr}}) = (\underline{\mathrm{apr}}_R, \overline{\mathrm{apr}}_R)$.

In the proof of the proposition it is stated which relation R we obtain, namely:

$$y \in R(x) \Leftrightarrow x \in \overline{\mathrm{apr}}(\{y\}). \tag{21}$$

The approximation operators from Table 6 which satisfy the conditions of Proposition 13 are the pairs 1–4, 9–11, 14, 16 and 17–22 [10].

Pairs 1–4, 9–11 and 17–22 are all element-based approximation operators of the form $(\underline{\mathrm{apr}}^{\mathbb{C}_j}_{N_i}, \overline{\mathrm{apr}}^{\mathbb{C}_j}_{N_i})$. Given a pair $(\underline{\mathrm{apr}}^{\mathbb{C}_j}_{N_i}, \overline{\mathrm{apr}}^{\mathbb{C}_j}_{N_i})$, the relation $R^{\mathbb{C}_j}_{N_i}$ obtained from this pair is given as follows:

$$y \in R^{\mathbb{C}_j}_{N_i}(x) \Leftrightarrow x \in \overline{\mathrm{apr}}^{\mathbb{C}_j}_{N_i}(\{y\}) \Leftrightarrow y \in N_i^{\mathbb{C}_j}(x). \tag{22}$$

In other words, $R^{\mathbb{C}_j}_{N_i}(x) = N_i^{\mathbb{C}_j}(x)$ for all $x \in U$.

For the pairs 14 and 16, we obtain the relation R_{H_3} and R_{H_5}, respectively

$$y \in R_{H_3}(x) \Leftrightarrow x \in H_3^{\mathbb{C}}(\{y\}) \Leftrightarrow x \in N_2^{\mathbb{C}}(y) \tag{23}$$

and

$$y \in R_{H_5}(x) \Leftrightarrow x \in H_5^{\mathbb{C}}(\{y\}) \Leftrightarrow x \in N_1^{\mathbb{C}}(y). \tag{24}$$

Based on Eq. (23), we can define the following neighborhood operator on U:

$$\forall x \in U : N_{H_3}(x) = R_{H_3}(x) = \{y \in U : x \in N_2^{\mathbb{C}}(y)\} \tag{25}$$

Therefore, we can rewrite the definition of $H_3^{\mathbb{C}}$ as follows: for $A \subseteq U$,

$$H_3^{\mathbb{C}}(A) = \bigcup\{N_2^{\mathbb{C}}(y) : y \in A\}$$

$$= \{x \in U : (\exists y \in A)(x \in N_2^{\mathbb{C}}(y)\}$$
$$= \{x \in U : (\exists y \in U)(y \in N_{H_3}(x) \wedge y \in A)\}$$
$$= \{x \in U : N_{H_3}(x) \cap A \neq \emptyset\}.$$

Hence, $H_3^{\mathbb{C}}$ is an element-based upper approximation operator, and by Proposition 9, the corresponding neighborhood operator N_{H_3} differs from each of the $N_i^{\mathbb{C}_j}$'s.

In a similar way, we can define a neighborhood operator based on Eq. (24):

$$\forall x \in U : N_{H_5}(x) = R_{H_5}(x) = \{y \in U : x \in N_1^{\mathbb{C}}(y)\} \tag{26}$$

Moreover, the approximation operator $H_5^{\mathbb{C}}$ can be written as follows:

$$\forall A \subseteq U : H_5^{\mathbb{C}}(A) = \{x \in U : N_{H_5}(x) \cap A \neq \emptyset\}.$$

Hence, we conclude that all approximation operators in Table 6 which satisfy the conditions of Proposition 13 are element-based approximation operators. In addition, all relations which are obtained by Proposition 13 are defined by the neighborhood operator used for the covering-based approximation operator. Moreover, we conclude that pairs 14 and 16 generate the inverse relation of the relations obtained by pairs 2 and 1, respectively, where the inverse relation of R is defined by $x \in R^{-1}(y) \Leftrightarrow y \in R(x)$: for $x, y \in U$,

$$y \in R_{H_3}(x) \Leftrightarrow x \in N_2^{\mathbb{C}}(y) \Leftrightarrow x \in R_{N_2}^{\mathbb{C}}(y),$$
$$y \in R_{H_5}(x) \Leftrightarrow x \in N_1^{\mathbb{C}}(y) \Leftrightarrow x \in R_{N_1}^{\mathbb{C}}(y).$$

As stated in Section 4.2, the dual pairs of approximation operators which yield the best accuracy, are the pairs 1 and 16. We see that these are exactly the pairs related to the approximation operators based on relation $R_{N_1}^{\mathbb{C}}$ and its inverse relation R_{H_5}.

5.2. Properties of the generated relations

In this section, we want to study which properties the relation R satisfies, if R is generated by a neighborhood operator N. The properties we consider are the following:

(a) The relation R is reflexive if and only if $\forall x \in U$: $x \in R(x)$.
(b) The relation R is symmetric if and only if $\forall x, y \in U$: $x \in R(y) \Rightarrow y \in R(x)$.
(c) The relation R is transitive if and only if $\forall x, y, z \in U$: $x \in R(y)$, $y \in R(z) \Rightarrow x \in R(z)$.

If the relation R is reflexive, symmetric and transitive, then R is an equivalence relation. In this case, the relation-based rough set model defined in Eqs. (19) and (20) coincides with the original model of Pawlak (Eqs. (1) and (2)).

To find out which properties the generated relation satisfies, it is sufficient to verify the properties of the neighborhood operator from which the relation R is generated.

Proposition 14. *Let N be a neighborhood operator and R and S the relations defined by*

$$\forall x, y \in U : y \in R(x) \Leftrightarrow y \in N(x)$$

and

$$\forall x, y \in U : y \in S(x) \Leftrightarrow x \in N(y),$$

i.e., S is the inverse relation of R, then

(a) *R is reflexive if and only if S is reflexive if and only if N is reflexive, i.e., $\forall x \in U$: $x \in N(x)$,*
(b) *R is symmetric if and only if S is symmetric if and only if N is symmetric, i.e.,*

$$\forall x, y \in U : x \in N(y) \Rightarrow y \in N(x).$$

In this case, $R = S$.

Moreover, if N is a reflexive neighborhood operator, then

(c) *R is transitive if and only if S is transitive if and only if N satisfies*

$$\forall x, y \in U : x \in N(y) \Rightarrow N(x) \subseteq N(y). \tag{27}$$

Proof. It is clear that (a) and (b) hold. For (c), we will prove it for R, the proof for S is similar. Let N be a reflexive neighborhood operator. If N satisfies Eq. (27), then for $x, y, z \in U$ with $x \in R(y)$ and $y \in R(z)$ we have that $x \in N(y)$ and $y \in N(z)$. Hence, $N(x) \subseteq N(y)$ and $N(y) \subseteq N(z)$ which means that $N(x) \subseteq N(z)$. Since N is reflexive, we have that $x \in N(z)$ and thus, $x \in R(z)$.

On the other hand, if the relation R is transitive, we can prove that N satisfies Eq. (27): let $x, y, z \in U$ with $x \in N(y)$ and $z \in N(x)$, then $x \in R(y)$ and $z \in R(x)$, hence, $z \in R(y)$. We conclude that $z \in N(y)$ and thus, $N(x) \subseteq N(y)$. □

In the following proposition, we study which properties $N_1^{\mathbb{C}}$, $N_2^{\mathbb{C}}$, $N_3^{\mathbb{C}}$ and $N_4^{\mathbb{C}}$ satisfy.

Proposition 15. *Let $\mathbb{C}$ be a covering of U.*

Table 7
Neighborhood operators based on $(U, \mathbb{C})$ for Example 15.

x	$N_1^{\mathbb{C}}(x)$	$N_2^{\mathbb{C}}(x)$	$N_3^{\mathbb{C}}(x)$	$N_4^{\mathbb{C}}(x)$
1	{1}	{1, 2, 3}	{1}	{1, 2, 3}
2	{1, 2}	{1, 2}	{1, 2}	{1, 2}
3	{3}	{3}	{1, 3}	{1, 3}

Table 8
Properties of relations induced by element-based approximation operators.

Pairs	N	Properties relation
1, 16, 19	$N_1^{\mathbb{C}}$	Reflexive and transitive
2, 14, 20, 22	$N_2^{\mathbb{C}}$	Reflexive
3, 17, 18, 21	$N_3^{\mathbb{C}}$	Reflexive and transitive
4, 9, 10, 11	$N_4^{\mathbb{C}}$	Reflexive and symmetric

(a) *The neighborhood operators* $N_1^{\mathbb{C}}$, $N_2^{\mathbb{C}}$, $N_3^{\mathbb{C}}$ *and* $N_4^{\mathbb{C}}$ *are reflexive.*
(b) *The neighborhood operator* $N_4^{\mathbb{C}}$ *is symmetric.*
(c) *The neighborhood operators* $N_1^{\mathbb{C}}$ *and* $N_3^{\mathbb{C}}$ *satisfy Eq. (27).*

Proof.

(a) The reflexivity of the four neighborhood operators follows immediately by definition.
(b) Take $x, y \in U$. Then

$$\begin{aligned} x \in N_4^{\mathbb{C}}(y) &\Leftrightarrow \exists K \in \mathcal{C}(\mathbb{C}, y) : x \in K \\ &\Leftrightarrow \exists K \in \mathcal{C}(\mathbb{C}, x) : y \in K \\ &\Leftrightarrow y \in N_4^{\mathbb{C}}(x). \end{aligned}$$

(c) Take $x, y \in U$ and assume $x \in N_1^{\mathbb{C}}(y)$. Take $z \in N_1^{\mathbb{C}}(x)$ and $K \in \mathrm{md}(\mathbb{C}, y)$. Since $x \in N_1^{\mathbb{C}}(y)$, $x \in K$. Thus, there exists a $K' \in \mathrm{md}(\mathbb{C}, x)$ with $K' \subseteq K$, and $z \in K'$ since $z \in N_1^{\mathbb{C}}(x)$. Hence, $z \in K$ and thus, $z \in N_1^{\mathbb{C}}(y)$.
Take $x, y \in U$ and assume $x \in N_3^{\mathbb{C}}(y)$. Take $z \in N_3^{\mathbb{C}}(x)$ and $K \in \mathrm{MD}(\mathbb{C}, y)$. Since $x \in N_3^{\mathbb{C}}(y)$, $x \in K$. Thus, there exists a $K' \in \mathrm{MD}(\mathbb{C}, x)$ with $K \subseteq K'$, and $z \in K'$ since $z \in N_3^{\mathbb{C}}(x)$. Since $K \in \mathrm{MD}(\mathbb{C}, y)$, $K = K'$. Hence, $z \in K$ and $z \in N_3^{\mathbb{C}}(y)$.

□

The neighborhood operators $N_1^{\mathbb{C}}$, $N_2^{\mathbb{C}}$ and $N_3^{\mathbb{C}}$ are not symmetric and the operators $N_2^{\mathbb{C}}$ and $N_4^{\mathbb{C}}$ do not satisfy Eq. (27). We illustrate this in Example 15.

Example 15. Let $U = \{1, 2, 3\}$ and $\mathbb{C} = \{\{3\}, \{1, 2\}, \{1, 3\}\}$.
From Table 7 it is easy to see that $N_1^{\mathbb{C}}$, $N_2^{\mathbb{C}}$ and $N_3^{\mathbb{C}}$ are not symmetric and that $N_2^{\mathbb{C}}$ and $N_4^{\mathbb{C}}$ do not satisfy Eq. (27).

We summarize the results for the generated relations derived from Propositions 14 and 15 in Table 8. The pairs 1–4, 9–11, 14, 16 and 17–22 all induce relation-based rough set approximation operators based on a reflexive relation. For pairs 4, 9, 10 and 11, the relation is also symmetric and for pairs 1, 3, 16, 17, 18, 19 and 21, the relation is also transitive. Note that none of the approximation operators is equivalent with Pawlak's model, since none of the generated relations is an equivalence relation.

In the next section, we discuss some ideas concerning applications.

6. Application to data analysis

As is commonly known, rough set theory has been applied extensively as a methodology for data analysis based on the approximation of concepts in information systems. It allows to infer data dependencies that are useful, among others, for feature selection and decision model construction (see e.g. [1]).

As an extension of classical rough sets, covering-based rough sets have also been considered for this kind of applications. In particular, the pioneering articles of Chen et al. [3] and Tsang et al. [18] led to algorithms for feature selection in decision systems involving covering-based rough sets. As their starting point, they use the lower approximation operator $\underline{\mathrm{apr}}_{N_1}^{\mathbb{C}}$, which corresponds to the neighborhood operator $N_1^{\mathbb{C}}$ (group A in Table 4). They use this approximation operator to define the positive region as the union of lower approximations of decision classes, construct the corresponding discernibility matrix, and develop methods to obtain decision reducts, analogously as for classical rough sets. More recently, Wang et al. [19] developed a heuristic algorithm to find a minimal subset of attributes that approximate an optimal reduct. Their method mainly reduces the computational complexity of the methods in [3]. On the other hand, Tan et al. [15] introduced matrix-based methods for computing approximations and reducts with covering-based rough sets. Using minimal and maximal descriptions which can be easily obtained by the matrix-based methods, a new discernibility matrix can be constructed for which the total number of non-empty discernibility sets could be dramatically reduced.

All of the above applications have in common that they focus on one particular definition of approximation operator, namely $\underline{\mathrm{apr}}^{\mathbb{C}}_{N_1}$. In this section, we argue that in this way, a large part of the potential of covering-based rough sets remains untapped, and that the consideration of other approximation operators, as well as the use of various approximation operators within a single application, can be beneficial.

In order to illustrate our ideas, we will work with a data set that emerges as a combination of various information systems, as was also considered in [3] and [18]. In particular, let U be a set of objects, and consider n ($n > 1$) partitions of U. Each of these partitions may be the result, for instance, of an expert categorizing the objects into sets of mutually indiscernible ones. Another way of obtaining the partitions could be to apply n different clustering algorithms to the data, again leading to n different data partitions. Once these have been obtained, we construct the covering $\mathbb{C}$ as the union of all the blocks appearing in the individual partitions.

Along with $\mathbb{C}$, we can construct the five derived coverings considered in this article:

1. $\mathbb{C}_1$ collects, for each object x in U, the minimal blocks K of $\mathbb{C}$ that contain x; in other words, those blocks of $\mathbb{C}$ that represent the most specific information about x, discerning it maximally from other objects.
2. $\mathbb{C}_2$ collects, for each object x in U, the maximal blocks K of $\mathbb{C}$ that contain x; in other words, those blocks of $\mathbb{C}$ that represent the least specific information about x, discerning it minimally from other objects.
3. $\mathbb{C}_3$ considers as basic information granules the neighborhoods $N_1^{\mathbb{C}}(x)$ of each object x in U, that is, those objects which are considered indiscernible from x according to each of the partitions. In [3], these objects make up the "default" neighborhood of x.
4. $\mathbb{C}_4$ brings together the neighborhoods $N_4^{\mathbb{C}}(x)$ of each object x in U, that is, those objects which are considered indiscernible from x according to at least one of the partitions. In [35], these objects are called the friends of x.
5. $\mathbb{C}_\cap$ contains those K in $\mathbb{C}$ that cannot be reconstructed as the intersection of other blocks of $\mathbb{C}$.

Combining the six coverings with the four neighborhood operators N_i, we obtain 13 different neighborhood operators, which form a partial order as discussed in Section 3. The smallest one among them is the default neighborhood operator $N_1^{\mathbb{C}}$. This neighborhood operator discerns two objects x and y if at least one expert considers them discernible, which implies a complete trust in the judgment of the experts, a condition which may not be desirable under all circumstances.

It is possible to weaken the definition of $N_1^{\mathbb{C}}$ in several ways. For instance, an object belongs to the neighborhood $N_3^{\mathbb{C}}(x)$ of x if it coincides with x in all of its maximal descriptions, but not necessarily in the non-maximal ones. Since $N_3^{\mathbb{C}} = N_1^{\mathbb{C}_2}$ holds, $y \in N_3^{\mathbb{C}}(x)$ means that according to the least specific information obtained by combining the n partitions, x and y are indiscernible. In terms of the experts, it means that if one expert assigns x and y to different blocks, there will always be another expert who assigns them to the same block, and that block also includes all objects the first expert considers indiscernible from x. In this way, we do not commit blindly to a single expert's opinion, but rather give precedence to more cautious experts when deciding whether x and y are to be considered discernible.

Similarly, the neighborhood $N_3^{\mathbb{C}_3}(x)$ is constructed using the default neighborhoods as our basic information granules (covering $\mathbb{C}_3$). In general, this neighborhood is not comparable to $N_3^{\mathbb{C}}$ as we saw in Section 3.

As another example, $y \in N_2^{\mathbb{C}}(x)$ means that y belongs to at least one minimal description of x, or also that at least one expert assigns x and y to the same block K of $\mathbb{C}$, and that none of the remaining experts assign x to a proper subblock K' of K. By the fact that $N_2^{\mathbb{C}} = N_2^{\mathbb{C}_1}$, this means that taking into account the most specific information obtained by combining the partitions, there is evidence that x and y are indiscernible. Note how this is a stronger requirement than $y \in N_4^{\mathbb{C}}(x)$, which means that x and y coincide in at least one block of $\mathbb{C}$, not necessarily minimal for x, i.e., one expert did not discern between x and y. Put differently, in this case we will only discern between x and y when all experts reach consensus that they should be discerned, while in the case of $N_2^{\mathbb{C}}$, we can reach the same conclusion (x and y discernible) even if some more cautious experts are not willing to separate them.

In a similar way, interpretations can be obtained for each of the other neighborhood operators, and a decision maker/data scientist can select the proper operator according to the needs of the application at hand.

Assume now that the data set we considered is converted into a decision system, that is, each object in U is assigned to a unique decision class ω in a set of classes Ω. An important consequence of the partial order of neighborhood operators $N_i^{\mathbb{C}_j}$ and their associated approximation operators is that there is also an associated partial order of positive regions, defined as, for x in U,

$$\mathrm{POS}^{\mathbb{C}_j}_{N_i} = \bigcup_{\omega \in \Omega} \underline{\mathrm{apr}}_{N_i^{\mathbb{C}_j}}(\omega).$$

In particular, the larger the neighborhood operator, the smaller the associated positive region. Indeed, an object x belongs to $\mathrm{POS}^{\mathbb{C}_j}_{N_i}$ if its entire neighborhood $N_i^{\mathbb{C}_j}(x)$ has the same class label.

The above observation also suggests a potential use of multiple neighborhood operators in a single data analysis application: for each object, we may consider the smallest positive region to which it belongs (if any). In this way, objects can be assigned different priority levels, which can be used in processes like rule induction, feature selection and instance selection. For instance, an object x for which the entire neighborhood $N_4^{\mathbb{C}_4}(x)$ belongs to the same decision class may be preferred over one for which only $N_1^{\mathbb{C}}(x)$ belongs to the same decision class.

7. Conclusion and future work

Neighborhood operators are a fundamental concept for the development of covering-based rough sets. In this article, we have undertaken a comprehensive study of such operators, as well as of the resulting approximation operators. Our investigations have led to the introduction of nine new neighborhood operators based on a covering, which complement the four basic ones considered in [27]. For the resulting 13 operators, we have established the lattice showing their partial order, i.e., expressing which operators give rise to smaller or larger element neighborhoods.

Next, we have shown that for three of the new neighborhood operators, the corresponding element-based pairs of approximation operators coincide with three of the 16 pairs considered in the framework of [10], while the remaining six ones result in new definitions. We have also integrated these six new pairs into the Hasse diagram showing the partial order of covering rough set approximation operators, i.e., expressing which operators give rise to smaller or larger set approximations.

Also, we have discussed the connection between the 22 pairs of covering-based approximation operators and the alternative framework of relation-based approximation operators. We have found that 15 out of the 22 pairs are equivalent with a pair of relation-based approximation operators. For each of the 15 pairs, we have derived the definition of the obtained relation, as well as the properties that this relation satisfies.

Finally, we have shown how the different neighborhood operators can be interpreted in the context of data analysis, and how they can be selected to suit the varying conditions that govern applications involving multiple sources of information (for instance, combining information provided by different experts, or by different clustering algorithms). Our findings also revealed that the existing work on feature selection by covering-based rough sets, discussed in Refs. [3,15,17,19], only focuses on one type of rough set approximation operator ($\underline{\mathrm{apr}}_{N_1}^{\mathbb{C}}$) which from the practical point of view may not always be the most convenient, since it supposes full trust in the different information sources. We believe that the new neighborhood operators and associated approximation operators we consider offer the data scientist/decision maker a wide and meaningful range of different trust options to choose from.

It is our hope that our investigations provide more insight into the foundations of covering-based rough sets. At the same time, we acknowledge that there still remain several avenues for further theoretical research in this direction. For example, the neighborhood operators $N_i^{\mathbb{C}_j}$ studied in this paper are not the only meaningful ones; a case in point are the neighborhood operators N_{H_3} from Eq. (25) and N_{H_5} from Eq. (26), which yield the upper approximation operators $H_3^{\mathbb{C}}$ and $H_5^{\mathbb{C}}$ proposed respectively by Tsang et al. in [17] and by Zhu in [36]. In general, future work includes the following directions:

- Study of the upper approximation operators $H_i^{\mathbb{C}_j}$, with $i = 1, 3, 4, 5$ and $\mathbb{C}_j$ one of the derived coverings obtained from an original covering $\mathbb{C}$.
- Study of the upper approximation operators obtained by generalizing $N_2^{\mathbb{C}}$ in $H_3^{\mathbb{C}}$ and $N_1^{\mathbb{C}}$ in $H_5^{\mathbb{C}}$ to an arbitrary neighborhood operator N, that is, for $A \subseteq U$, $H(A) = \bigcup\{N(x) : x \in A\}$.
- Study of the relation-based approximation operators obtained from the inverse relation corresponding to one of the non-symmetric neighborhood operators $N_1^{\mathbb{C}_j}$, $N_2^{\mathbb{C}_j}$ and $N_3^{\mathbb{C}_j}$.
- Study of the covering-based rough set approximation operators $\underline{C}_3$, $\overline{C}_3$ and $\overline{C}^{\%}$ defined in [12,13], which were not considered in our framework. A comparison between the operators in [12,13] considered in this article is presented in Appendix B.

Besides the theoretical study of new neighborhood and approximation operators, an essential research direction will include the practicality of all the approximation operators in real-world applications. Therefore, experimental analysis of the operators on larger datasets will be an important study. A particular application we are interested in is feature selection. In [5], an extension of the IRBASIR algorithm from [6] was proposed. IRBASIR is an algorithm to derive decision rules from a, possibly incomplete, decision table. An important part of the extension of [5] is the incorporation of a feature selection procedure. The proposed reduction algorithm depends on the definition of a quality measure, which is itself based on two neighborhood operators. A downside is that these operators require the specification of many parameters, such as the threshold above which instances are considered sufficiently similar and the weights of individual attributes in the similarity calculations. In their experimental study, the accuracy obtained by their complete classification algorithm was shown to be significantly better than previous proposals. However, these experiments were conducted on a small set of only 15 datasets and the chosen parameters might not be optimal in general. Our proposed neighborhood operators form valid replacement candidates to be used internally in this method, as they require no further parameter tuning.

Although many research considering relation-based approximation operators is done in this area, we have seen that not all covering-based approximation operators fit in this relation-based framework. For example, the operators defined in pairs 5, 12 and 13 have a very good accuracy, but they are not equivalent to relation-based operators. A related problem is to generate meaningful coverings based on given data. This research can provide a new perspective on the application of rough sets in machine learning.

Table A9
Negative results about the partial order between approximation operators, obtained from Corollary 6 and Fig. 1.

	Pair 17	Pair 18	Pair 19	Pair 21	Pair 22
Pair 5			$5 \not\leq 19$		$5 \not\leq 22$
		$18 \not\leq 5$			
Pair 6					$6 \not\leq 22$
	$17 \not\leq 6$	$18 \not\leq 6$			
Pair 7					$7 \not\leq 22$
	$17 \not\leq 7$	$18 \not\leq 7$			
Pair 8			$8 \not\leq 19$		$8 \not\leq 22$
	$17 \not\leq 8$	$18 \not\leq 8$			
Pair 12	$12 \not\leq 17$	$12 \not\leq 18$	$12 \not\leq 19$	$12 \not\leq 21$	$12 \not\leq 22$
	$17 \not\leq 12$	$18 \not\leq 12$	$19 \not\leq 12$	$21 \not\leq 12$	$22 \not\leq 12$
Pair 13	$13 \not\leq 17$	$13 \not\leq 18$	$13 \not\leq 19$	$13 \not\leq 21$	$13 \not\leq 22$
		$18 \not\leq 13$			
Pair 14	$14 \not\leq 17$	$14 \not\leq 18$	$14 \not\leq 19$	$14 \not\leq 21$	$14 \not\leq 22$
		$18 \not\leq 14$			
Pair 15	$15 \not\leq 17$	$15 \not\leq 18$	$15 \not\leq 19$	$15 \not\leq 21$	$15 \not\leq 22$
	$17 \not\leq 15$	$18 \not\leq 15$			
Pair 16	$16 \not\leq 17$	$16 \not\leq 18$	$16 \not\leq 19$	$16 \not\leq 21$	$16 \not\leq 22$

Table A10
Lower approximations for the covering approximation space $(U, \mathbb{C})$ from Example 16.

	{1, 3}	{1, 4}	{2, 4}	{3, 4}	{1, 3, 4}
Pair 5	{1, 3}	{1, 4}	{2, 4}	{3, 4}	{1, 3, 4}
Pair 6	{1, 3}	{1, 4}	∅	∅	{1, 3, 4}
Pair 7	∅	∅	∅	∅	{1, 3, 4}
Pair 8	{1, 3}	{1, 4}	{2, 4}	{3, 4}	{1, 3, 4}
Pair 13	{1, 3}	{1}	{2, 4}	{3}	{1, 3}
Pair 14	∅	{1}	{2, 4}	{3}	{1, 3}
Pair 15	{1, 3}	{1}	{2, 4}	∅	{1}
Pair 16	{1, 3}	{1}	{2, 4}	{3}	{1, 3}
Pair 17	∅	{1, 4}	{2, 4}	{3, 4}	{1, 3, 4}
Pair 18	{1, 3}	{1}	{2, 4}	{3}	{1, 3}
Pair 19	∅	∅	∅	{3, 4}	{1, 3, 4}
Pair 22	∅	∅	{2}	∅	{1, 3, 4}

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Appendix A. Counterexamples regarding Section 4.2

We discuss the counterexamples to obtain Fig. 3 in Section 4.2. Note that the discussion for pair 20 was fully stated in Section 4.2. With the same argument as in Section 4.2 for the pairs 12 and 20, we can obtain the results in Table A.9, which lists order relations that cannot hold, since they are in conflict with previous results stated in [10]. For example, if $17 \leq 6$, then by the fact that $9 \leq 17$ (see Corollary 6), pair 9 would be comparable with pair 6, which is a contradiction.

To continue the study of comparability of pairs 17, 18, 19, 21 and 22, we consider the following example:

Example 16. Let $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1\}, \{3\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{1, 4\}, \{2, 3, 4\}\}$. We calculate the lower approximation of some subsets of U by different approximation operators in Table A.10.

A1. Pair 17

From Table A.9, we conclude that pair 17 is incomparable with pairs 12 and 15. Furthermore, $17 \leq 6$, $17 \leq 7$, $17 \leq 8$ and $13 \leq 17$, $14 \leq 17$, $16 \leq 17$ do not hold. From Table A.10, we conclude that $5 \not\leq 17$, $6 \not\leq 17$, $8 \not\leq 17$ ($A = \{1, 3\}$) and $17 \not\leq 13$, $17 \not\leq 14$, $17 \not\leq 16$ ($A = \{3, 4\}$). Hence, pair 17 is incomparable with pairs 6, 8, 13, 14 and 16. The only two possible order relations are $17 \leq 5$ and $7 \leq 17$. Both do not hold as illustrated in the next examples.

Example 17. Let $U = \{1, 2, 3\}$ and $\mathbb{C} = \{\{1, 2\}, \{2, 3\}\}$, then $\underline{\text{apr}}'_{\mathbb{C}}(\{2\}) = \emptyset$ and $\underline{\text{apr}}_{N_3}^{\mathbb{C}_1}(\{2\}) = \{2\}$. Hence, $17 \not\leq 5$.

Example 18. Let $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 4\}\}$, then $\underline{\mathrm{apr}}'_{\mathbb{C}_4}(\{1, 2, 4\}) = \{1, 2, 4\}$ and $\underline{\mathrm{apr}}^{\mathbb{C}_1}_{N_3}(\{1, 2, 4\}) = \{1, 4\}$. Hence, $7 \not\leq 17$.

We conclude that pair 17 is also incomparable with pair 5 and pair 7.

A2. Pair 18

From Table A.9, we obtain that pair 18 is incomparable with pairs 12–15. Furthermore, $18 \leq 5$, $18 \leq 6$, $18 \leq 7$, $18 \leq 8$ and $16 \leq 18$ do not hold. From Table A.10, we derive that $5 \not\leq 18$, $6 \not\leq 18$, $8 \not\leq 18$ ($A = \{1, 4\}$), hence pair 18 is incomparable with pairs 5, 6 and 8. The only relations that can hold are $7 \leq 18$ and $18 \leq 16$. However, both do not hold as illustrated in the following example:

Example 19. Let $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1\}, \{2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 3, 4\}\}$, then $\underline{\mathrm{apr}}'_{\mathbb{C}_4}(\{2, 4\}) = \{2, 4\}$ and $\underline{\mathrm{apr}}^{\mathbb{C}_3}_{N_3}(\{2, 4\}) = \{2\}$. Hence, $7 \not\leq 18$. Moreover, if $\mathbb{C} = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 4\}\}$, then $(H^{\mathbb{C}}_5)^{\partial}(\{1, 3, 4\}) = \{3, 4\}$ and $\underline{\mathrm{apr}}^{\mathbb{C}_3}_{N_3}(\{1, 3, 4\}) = \{1, 3\}$. Hence, $18 \not\leq 16$.

We conclude that pair 18 is incomparable with pairs 7 and 16.

A3. Pair 19

In Section 4.2 it is stated that pair 19 is comparable with pair 7. From Table A.9 it is clear that pair 19 is incomparable with pair 12 and that $5 \leq 19$, $8 \leq 19$ and $13 \leq 19$, $14 \leq 19$, $15 \leq 19$, $16 \leq 19$ do not hold. From Table A.10, we obtain that $19 \not\leq 13$, $19 \not\leq 14$, $19 \not\leq 15$, $19 \not\leq 16$ ($A = \{3, 4\}$), hence, pair 19 is incomparable with pairs 13–16. As illustrated in the following example, $19 \leq 5$, $19 \leq 6$ and $19 \leq 8$ do not hold.

Example 20. Let $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}$, then $\underline{\mathrm{apr}}^{\mathbb{C}_4}_{N_1}(\{1\}) = \{1\}$ and $\underline{\mathrm{apr}}'_{\mathbb{C}}(\{1\}) = \underline{\mathrm{apr}}'_{\mathbb{C}_2}(\{1\}) = \underline{\mathrm{apr}}'_{\mathbb{C}_\cap}(\{1\}) = \emptyset$.

Thus, pair 19 is incomparable with pairs 5 and 8. Moreover, $6 \not\leq 19$.

Example 21. Let $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$, then $\underline{\mathrm{apr}}^{\mathbb{C}_4}_{N_1}(\{1, 2\}) = \emptyset$ and $\underline{\mathrm{apr}}'_{\mathbb{C}_2}(\{1, 2\}) = \{1, 2\}$.

Hence, pair 19 is incomparable with pair 6.

A4. Pair 21

From Table A.9, we obtain that pair 21 is incomparable with pair 12 and that $13 \leq 21$, $14 \leq 21$, $15 \leq 21$, $16 \leq 21$ do not hold. The following example illustrates that $21 \not\leq 13$, $21 \not\leq 14$, $21 \not\leq 15$, $21 \not\leq 16$:

Example 22. Let $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1\}, \{2\}, \{1, 2\}, \{1, 4\}, \{3, 4\}\}$, then $\underline{\mathrm{apr}}^{\mathbb{C}_4}_{N_3}(\{1, 2, 4\}) = \{1, 2, 4\}$ and $(H^{\mathbb{C}}_1)^{\partial}(\{1, 2, 4\}) = (H^{\mathbb{C}}_3)^{\partial}(\{1, 2, 4\}) = (H^{\mathbb{C}}_4)^{\partial}(\{1, 2, 4\}) = (H^{\mathbb{C}}_5)^{\partial}(\{1, 2, 4\}) = \{1, 2\}$.

We conclude that pair 21 is incomparable with pairs 13–16. Next, we study whether pair 21 is comparable with pairs 5–8. Due to the next example, it is clear that pair 21 is incomparable with them.

Example 23. Let $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$, then $\underline{\mathrm{apr}}^{\mathbb{C}_4}_{N_3}(\{1, 2, 3\}) = \emptyset$ and $\underline{\mathrm{apr}}'_{\mathbb{C}}(\{1, 2, 3\}) = \underline{\mathrm{apr}}'_{\mathbb{C}_2}\{1, 2, 3\}) = \underline{\mathrm{apr}}'_{\mathbb{C}_4}(\{1, 2, 3\}) = \underline{\mathrm{apr}}'_{\mathbb{C}_\cap}(\{1, 2, 3\}) = \{1, 2, 3\}$. Hence, $5 \not\leq 21$, $6 \not\leq 21$, $7 \not\leq 21$ and $8 \not\leq 21$. Moreover, if $\mathbb{C} = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}$, then $\underline{\mathrm{apr}}^{\mathbb{C}_4}_{N_3}(\{1\}) = \{1\}$ and $\underline{\mathrm{apr}}'_{\mathbb{C}}(\{1\}) = \underline{\mathrm{apr}}'_{\mathbb{C}_2}\{1\}) = \underline{\mathrm{apr}}'_{\mathbb{C}_4}(\{1\}) = \underline{\mathrm{apr}}'_{\mathbb{C}_\cap}(\{1\}) = \emptyset$. Hence, $21 \not\leq 5$, $21 \not\leq 6$, $21 \not\leq 7$ and $21 \not\leq 8$.

A5. Pair 22

In Section 4.2 it is stated that pair 22 is comparable with pairs 5 and 8. From Table A.9, we obtain that pair 22 is incomparable with pair 12 and that none of pairs 6, 7 and 13–16 are smaller than 22. From Table A.10, we derive that $22 \not\leq 6$ and $22 \not\leq 7$ ($A = \{2, 4\}$) and that $22 \not\leq 13$, $22 \not\leq 14$, $22 \not\leq 15$ and $22 \not\leq 16$ ($A = \{1, 3, 4\}$). Hence, pair 22 is incomparable with pairs 6, 7 and 13–16.

Appendix B. Comparison with operators of [12,13]

In Table B.11, the comparison is given between the operators discussed in [12,13] and the operators discussed in this article. Operators $\underline{C}_3$, $\overline{C}_3$ and $\overline{C}^{\%}$ of [12,13] are not discussed in this article.

Table B11
Comparison of our framework and the framework of Samanta and Chakraborty.

Operator [12,13]	Equivalent operator	Operator [12,13]	Equivalent operator
$\underline{P}_1$	$\underline{\mathrm{apr}}^{\mathbb{C}}_{N_4}$	$\underline{C}_*$	$\underline{\mathrm{apr}}'_{\mathbb{C}}$
$\overline{P}_1$	$\overline{\mathrm{apr}}^{\mathbb{C}}_{N_4}$	$\overline{C}_*$	$H_1^{\mathbb{C}}$
$\underline{P}_2$	$\underline{\mathrm{apr}}'_{\mathbb{C}_4}$	$\underline{C}_-$	$\underline{\mathrm{apr}}'_{\mathbb{C}}$
$\overline{P}_2$	$\overline{\mathrm{apr}}'_{\mathbb{C}_4}$	$\overline{C}^-$	$\overline{\mathrm{apr}}''_{\mathbb{C}}$
$\underline{P}_3$	$\underline{\mathrm{apr}}'_{\mathbb{C}}$	$\underline{C}_\#$	$\underline{\mathrm{apr}}'_{\mathbb{C}}$
$\overline{P}_3$	$\overline{\mathrm{apr}}'_{\mathbb{C}}$	$\overline{C}^\#$	$H_3^{\mathbb{C}}$
$\underline{P}_4$	$\underline{\mathrm{apr}}$ (Pawlak)	$\underline{C}_@$	$\underline{\mathrm{apr}}'_{\mathbb{C}}$
$\overline{P}_4$	$\overline{\mathrm{apr}}$ (Pawlak)	$\overline{C}^@$	$H_4^{\mathbb{C}}$
$\underline{C}_1$	$\underline{\mathrm{apr}}'_{\mathbb{C}}$	$\underline{C}_+$	$\underline{\mathrm{apr}}'_{\mathbb{C}}$
$\overline{C}_1$	$\overline{\mathrm{apr}}'_{\mathbb{C}}$	$\overline{C}^+$	$H_5^{\mathbb{C}}$
$\underline{C}_2$	$\underline{\mathrm{apr}}^{\mathbb{C}}_{N_1}$	$\underline{C}_\%$	$\underline{\mathrm{apr}}'_{\mathbb{C}}$
$\overline{C}_2$	$\overline{\mathrm{apr}}^{\mathbb{C}}_{N_1}$	$\overline{C}^\%$	$\times$
$\underline{C}_3$	$\times$	Gr_*	$\underline{\mathrm{apr}}'_{\mathbb{C}}$
$\overline{C}_3$	$\times$	Gr^*	$\overline{\mathrm{apr}}''_{\mathbb{C}}$
$\underline{C}_4$	$\underline{\mathrm{apr}}^{\mathbb{C}_3}_{N_4}$	$\overline{C}^{Gr}$	$\overline{\mathrm{apr}}'_{\mathbb{C}}$
$\overline{C}_4$	$\overline{\mathrm{apr}}^{\mathbb{C}_3}_{N_4}$		
$\underline{C}_5$	$(H_5^{\mathbb{C}})^\partial$		
$\overline{C}_5$	$H_5^{\mathbb{C}}$		

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