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Notes on covering-based rough sets from topological point of view: Relationships with general framework of dual approximation operators



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ABSTRACT

In the recent article “On some types of covering rough sets from topological points of view” [14], the author develops a topological approach to covering-based rough sets. In this context, a number of corresponding approximation operators are introduced, their inclusion relationships are verified, and various conditions under which the operators coincide are proven.

On the other hand, a lot of effort has recently been dedicated by several authors to study covering-based approximation operators within a general framework of dual approximation operators [2,3,7–10,12].

In this note, we study correspondences between the framework of Zhao and the framework established in [2]. In particular, we evaluate how the newly introduced topological approximation operators relate to existing ones in terms of equalities and partial order relations.

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1. Introduction and preliminaries

Over the past two decades, numerous definitions of covering-based rough set approximations have been proposed in literature, see e.g. [1,8–10,13,15–17]. Yao and Yao [12] were the first who attempted to categorize them within a general framework of dual approximation operators, and their efforts were later extended in e.g. [2,3,6,7]. Apart from establishing equalities between the different proposals, a pertinent research question has been to order approximation operators in order to reach conclusions on their approximation accuracy (i.e., the ratio between the lower and the upper approximation) [3,7]. Indeed, from a practical perspective, it is reasonable to consider the most accurate pair of operators, as the approximations will be the closest to the approximated set.

Zhao [14] recently introduced a number of covering-based approximation operators inspired by a topological approach. This paper aims to relate those operators to the general framework, making it clear which operators coincide with existing ones, and how the remaining ones fit in the Hasse diagram representing the partial order of approximation operators.

We start with discussing the basic notions of covering-based rough sets and provide both the general and topological framework of approximation operators.

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1.1. Basic notions of covering-based rough sets

Throughout this work we assume that the universe U is a non-empty set. In Pawlak's rough set model [5], given an equivalence relation E on U and a set A in U ,

$$\underline{\text{apr}}(A) = \{x \in U : [x]_E \subseteq A\} = \bigcup \{[x]_E \in U/E : [x]_E \subseteq A\}, \tag{1}$$

$$\overline{\text{apr}}(A) = \{x \in U : [x]_E \cap A \neq \emptyset\} = \bigcup \{[x]_E \in U/E : [x]_E \cap A \neq \emptyset\}, \tag{2}$$

are called the lower and upper approximation of A , respectively. The equivalence class $[x]_E = \{y \in U : (x, y) \in E\}$ of an object x describes all the objects of the universe which are indiscernible from x by E . The ordered pair (U, E) is called a Pawlak approximation space. A generalization of Pawlak's model can be obtained by replacing the partition U/E with a covering of U .

Definition 1. [15] Let $\mathbb{C} = \{K_i \subseteq U : i \in I\}$ be a family of non-empty subsets of U , with I a set of indices. \mathbb{C} is called a covering of U if $\bigcup_{i \in I} K_i = U$. The ordered pair (U, \mathbb{C}) is called a covering approximation space.

Another possibility is generalizing equivalence classes to neighborhoods. A neighborhood operator [12] is a mapping $N: U \rightarrow \mathcal{P}(U)$ which is generally assumed to be reflexive, i.e., $\forall x \in U : x \in N(x)$. Other possible properties which N can have include symmetry, i.e., $\forall x, y \in U : x \in N(y) \Leftrightarrow y \in N(x)$, and transitivity, i.e., $\forall x, y, z \in U : x \in N(y) \wedge y \in N(z) \Rightarrow x \in N(z)$, or equivalently, $\forall x, y \in U : x \in N(y) \Rightarrow N(x) \subseteq N(y)$.

Furthermore, given a neighborhood operator N , we define its inverse neighborhood operator N^{-1} by $x \in N^{-1}(y)$ if and only if $y \in N(x)$ for $x, y \in U$. It is clear that if N is symmetric, $N^{-1} = N$ (see [2]).

We also propose a partial order relation on the set of neighborhood operators on U . Let N and N' be two neighborhood operators on U , then we write $N \leq N'$ if and only if $\forall x \in U : N(x) \subseteq N'(x)$.

Some of the more prominent neighborhood operators are defined in terms of the minimal and maximal descriptions of an element x in U .

Definition 2. Let $x \in U$ and \mathbb{C} a covering on U . The minimal description [1] of x and maximal description [17] of x in \mathbb{C} are defined by,

$$\text{md}(\mathbb{C}, x) = \{K \in \mathcal{C}(\mathbb{C}, x) : (\forall S \in \mathcal{C}(\mathbb{C}, x))(S \subseteq K \Rightarrow K = S)\}, \tag{3}$$

$$\text{MD}(\mathbb{C}, x) = \{K \in \mathcal{C}(\mathbb{C}, x) : (\forall S \in \mathcal{C}(\mathbb{C}, x))(S \supseteq K \Rightarrow K = S)\}, \tag{4}$$

where $\mathcal{C}(\mathbb{C}, x) = \{K \in \mathbb{C} : x \in K\}$.

Yao and Yao [12] used the above constructs to define four neighborhood operators based on the covering \mathbb{C} :

$$N_1^{\mathbb{C}}(x) = \bigcap \{K \in \mathbb{C} : K \in \text{md}(\mathbb{C}, x)\} = \bigcap \mathcal{C}(\mathbb{C}, x), \quad N_2^{\mathbb{C}}(x) = \bigcup \{K \in \mathbb{C} : K \in \text{md}(\mathbb{C}, x)\},$$

$$N_3^{\mathbb{C}}(x) = \bigcap \{K \in \mathbb{C} : K \in \text{MD}(\mathbb{C}, x)\}, \quad N_4^{\mathbb{C}}(x) = \bigcup \{K \in \mathbb{C} : K \in \text{MD}(\mathbb{C}, x)\} = \bigcup \mathcal{C}(\mathbb{C}, x).$$

It is easy to check that all of them are reflexive. Moreover, $N_1^{\mathbb{C}}$ and $N_3^{\mathbb{C}}$ are transitive, and $N_4^{\mathbb{C}}$ is symmetric [3]. Furthermore, Yao and Yao also derived four new coverings from the original covering \mathbb{C} :

$$\mathbb{C}_1 = \{K \in \mathbb{C} : (\exists x \in U)(K \in \text{md}(\mathbb{C}, x))\}, \quad \mathbb{C}_2 = \{K \in \mathbb{C} : (\exists x \in U)(K \in \text{MD}(\mathbb{C}, x))\}$$

$$\mathbb{C}_3 = \{N_1^{\mathbb{C}}(x) : x \in U\}, \quad \mathbb{C}_4 = \{N_4^{\mathbb{C}}(x) : x \in U\}.$$

Additionally, they considered the intersection reduct \mathbb{C}_n of \mathbb{C} , defined by

$$\mathbb{C}_n = \mathbb{C} \setminus \left\{ K \in \mathbb{C} : (\exists \mathbb{C}' \subseteq \mathbb{C} \setminus \{K\}) \left(K = \bigcap \mathbb{C}' \right) \right\}.$$

It is possible to combine the four basic neighborhood operators with each of the five additional coverings to derive additional neighborhood operators $N_i^{\mathbb{C}_j}$, with $i \in \{1, 2, 3, 4\}$ and $\mathbb{C}_j \in \{\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4, \mathbb{C}_n\}$. In [3], it was shown that these 20 combinations together with the original four neighborhood operators based on the covering \mathbb{C} give rise to exactly 13 groups of equal neighborhood operators. If we also take into account their inverses, we obtain 22 different groups. They are denoted by the letters $A-M$, and appear in the last column of Table 2. Note that the operators G, H, J and M are symmetric, hence, they are equal to their respective inverse neighborhood operator.

1.2. General framework of dual approximation operators

In a covering approximation space, various definitions to shape the lower and upper approximation operators have been proposed. In this paper, we always consider dual pairs $(\underline{\text{apr}}, \overline{\text{apr}})$ of $\mathcal{P}(U) \rightarrow \mathcal{P}(U)$ mappings, i.e., $\underline{\text{apr}} = (\overline{\text{apr}})^\theta$ and $\overline{\text{apr}} = (\underline{\text{apr}})^\theta$, where for each $A \in \mathcal{P}(U)$,

$$(\overline{\text{apr}})^\theta(A) = \text{co}(\overline{\text{apr}}(\text{co } A)) \text{ and } (\underline{\text{apr}})^\theta(A) = \text{co}(\underline{\text{apr}}(\text{co } A)),$$

and co represents the set-theoretical complement.

The partial order relation on approximation operators \leq was considered in [7] and is defined as follows: let apr_1 and apr_2 be two approximation operators, then

$$\text{apr}_1 \leq \text{apr}_2 \Leftrightarrow \forall A \subseteq U: \text{apr}_1(A) \subseteq \text{apr}_2(A). \tag{5}$$

It is clear that for two dual pairs $(\underline{\text{apr}}_1, \overline{\text{apr}}_1)$ and $(\underline{\text{apr}}_2, \overline{\text{apr}}_2)$ it holds that

$$\overline{\text{apr}}_1 \leq \overline{\text{apr}}_2 \Leftrightarrow \underline{\text{apr}}_2 \leq \underline{\text{apr}}_1. \tag{6}$$

Hence, in studying dual pairs of approximation operators, it is sufficient to consider e.g. only the upper approximations. The overview of covering-based upper approximation operators discussed in this work can be found in Table 2.

Three main classes of approximation operators exist, which can be characterized as element-based, granule-based and system-based definitions, respectively [12].

Definition 3. [12] Let N be a neighborhood operator. The element-based lower and upper approximation operators corresponding to N are defined by, for $A \subseteq U$,

$$\underline{\text{apr}}_N(A) = \{x \in U : N(x) \subseteq A\}, \quad \overline{\text{apr}}_N(A) = \{x \in U : N(x) \cap A \neq \emptyset\}. \tag{7}$$

Representatives of the element-based definition can be obtained by using any of the neighborhood operators $N_i^{C_j}$ or $(N_i^{C_j})^{-1}$ discussed above. The following two propositions recall some relationships between neighborhood operators and the approximation operators they generate.

Proposition 1. Let N and N' be two neighborhood operators on U . Then $N \leq N' \Leftrightarrow \overline{\text{apr}}_N \leq \overline{\text{apr}}_{N'}$.

Proof. In [7], it was proven that $N \leq N' \Rightarrow \overline{\text{apr}}_N \leq \overline{\text{apr}}_{N'}$. On the other hand, let $y \in U$ such that $y \in N(x)$, then $N(x) \cap \{y\} \neq \emptyset$, thus $x \in \overline{\text{apr}}_N(\{y\})$. Hence, $x \in \overline{\text{apr}}_{N'}(\{y\})$ and $N'(x) \cap \{y\} \neq \emptyset$. We conclude that $y \in N'(x)$. \square

Corollary 1. [3] Let N and N' be two neighborhood operators on U . Then $N = N' \Leftrightarrow \overline{\text{apr}}_N = \overline{\text{apr}}_{N'}$.

The following proposition due to Yao [11], rephrased here to our terminology, provides a characterization of element-based approximation operators.

Proposition 2. [11] Assume $(\underline{\text{apr}}, \overline{\text{apr}})$ is a dual pair of approximation operators on U , and I an arbitrary index set. There exists a neighborhood operator N such that $(\underline{\text{apr}}, \overline{\text{apr}}) = (\underline{\text{apr}}_N, \overline{\text{apr}}_N)$ if and only if $\overline{\text{apr}}(\emptyset) = \emptyset$ and for all $A_i \subseteq U$, $i \in I$, $:\overline{\text{apr}}\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} \overline{\text{apr}}(A_i)$, i.e., $\overline{\text{apr}}$ is a complete join morphism. The neighborhood operator N is given by, for x, y in U ,

$$y \in N(x) \Leftrightarrow x \in \overline{\text{apr}}(\{y\}). \tag{8}$$

Definition 4. [12] The granule-based lower and upper approximation operators corresponding to a covering \mathbb{C} are defined by, for $A \subseteq U$,

$$\underline{\text{apr}}'_\mathbb{C}(A) = \bigcup \{K \in \mathbb{C} : K \subseteq A\}, \quad \overline{\text{apr}}'_\mathbb{C}(A) = (\underline{\text{apr}}'_\mathbb{C})^\theta(A). \tag{9}$$

Additional instances of granule-based approximation operators can be obtained by replacing the covering \mathbb{C} with any of the derived coverings discussed in the previous section.

Definition 5. [12] A family of subsets of U is called a closure system over U if it contains U and is closed under set intersection. Given a closure system \mathbb{S} over U , its dual system is defined by $\mathbb{S}' = \{\text{co } K : K \in \mathbb{S}\}$.

Definition 6. [12] Let $S = (S', \mathbb{S})$, where \mathbb{S} is a closure system. The system-based lower and upper approximation operator corresponding to S are defined by, for $A \subseteq U$,

$$\underline{\text{apr}}_S(A) = \bigcup \{K \in S' : K \subseteq A\}, \quad \overline{\text{apr}}_S(A) = \bigcap \{K \in \mathbb{S} : K \supseteq A\}. \quad (10)$$

Two representatives of the system-based definition given in [12] are $S_{\cap} = ((S_{\cap, \mathbb{C}})', S_{\cap, \mathbb{C}})$ and $S_{\cup} = (S_{\cup, \mathbb{C}}, (S_{\cup, \mathbb{C}})')$, where the intersection closure $S_{\cap, \mathbb{C}}$ of a covering \mathbb{C} is the minimal subset of $\mathcal{P}(U)$ that contains \mathbb{C} , \emptyset and U , and is closed under set intersection, and the union closure $S_{\cup, \mathbb{C}}$ of \mathbb{C} is the minimal subset of $\mathcal{P}(U)$ that contains \mathbb{C} , \emptyset and U , and is closed under set union.

Finally, we recall two pairs of approximation operators that do not correspond to any of the above-defined categories [3]. Their upper approximation operators were introduced by Żakowski [13] and Zhu [16], respectively.

Definition 7. Let (U, \mathbb{C}) be a covering approximation space, then for $A \subseteq U$,

$$\underline{\text{apr}}_{\mathbb{C}}^z(A) = (\overline{\text{apr}}_{\mathbb{C}}^z)^{\partial}(A), \quad \overline{\text{apr}}_{\mathbb{C}}^z(A) = \underline{\text{apr}}'_{\mathbb{C}}(A) \cup \left(\bigcup \left\{ \bigcup \text{md}(\mathbb{C}, x) : x \in A \setminus \underline{\text{apr}}'_{\mathbb{C}}(A) \right\} \right), \quad (11)$$

$$\underline{\text{apr}}_{\mathbb{C}}^{\text{zhu}}(A) = (\overline{\text{apr}}_{\mathbb{C}}^{\text{zhu}})^{\partial}(A), \quad \overline{\text{apr}}_{\mathbb{C}}^{\text{zhu}}(A) = \underline{\text{apr}}'_{\mathbb{C}}(A) \cup \left(\bigcup \left\{ K \in \mathbb{C} : K \cap (A \setminus \underline{\text{apr}}'_{\mathbb{C}}(A)) \neq \emptyset \right\} \right). \quad (12)$$

1.3. Topological approach to covering-based rough sets

We start by recalling some essential topological concepts that will be used in the remainder.

Definition 8. [4] A topology on a set U is a collection \mathcal{T} of subsets of U having the following properties:

1. \emptyset and U are in \mathcal{T} .
2. The union of the elements of any subcollection of \mathcal{T} is in \mathcal{T} .
3. The intersection of the elements of any finite subcollection of \mathcal{T} is in \mathcal{T} .

The ordered pair (U, \mathcal{T}) is called a topological space. The set $A \subseteq U$ is said to be open in U if it belongs to the collection \mathcal{T} , and closed in U if $\text{co} A$ is open in U . The interior $\text{int} A$ of A is the union of all open sets in U contained in A . The closure \overline{A} of A is the intersection of all closed sets in U containing A .

Definition 9. [4] Let (U, \mathcal{T}) be a topological space. If $V \subseteq U$, then (V, \mathcal{T}_V) is called a topological subspace of (U, \mathcal{T}) , where $\mathcal{T}_V = \{V \cap A : A \in \mathcal{T}\}$.

Definition 10. [4] Let (U, \mathcal{T}) be a topological space. A separation of U is a pair (A, B) of disjoint non-empty open sets in U whose union is U . The space (U, \mathcal{T}) is said to be connected if there does not exist a separation of U .

Definition 11. [4] Let (U, \mathcal{T}) be a topological space, x, y in U . We define an equivalence relation on U by setting $x \sim y$ if there is a connected subspace of (U, \mathcal{T}) containing both x and y . For each x in U , the equivalence class of x is called the component of x and is denoted by $[x]_{\sim}$.

Zhao [14] introduced the topology induced by \mathbb{C} , and showed that this notion indeed satisfies the conditions of Definition 8.

Definition 12. [14] Let (U, \mathbb{C}) be a covering approximation space. The topology \mathcal{T} on U induced by \mathbb{C} is defined as follows: $A \subseteq U$ is open in U if and only if for each x in A , there exist a subset $\{C_1, C_2, \dots, C_n\}$ of \mathbb{C} such that $x \in \bigcap_{i=1}^n C_i \subseteq A$.

Zhao [14] considered the following covering-based approximation operators for the covering-induced topology¹:

Definition 13. [14] Let (U, \mathbb{C}) be a covering approximation space. Given the topology \mathcal{T} on U induced by \mathbb{C} , the following lower and upper approximation operators are defined by, for $A \subseteq U$,

¹ For completeness, we mention that Zhao introduced the operators for finite universes only; however, nothing prevents us from considering these operators more generally in arbitrary universes, since all results about them used in this paper remain valid in the infinite case as well, as stated also by Zhao in his paper.

$$I^-(A) = \{x \in U : N_1^C(x) \subseteq A\} = \text{int } A, \quad I^+(A) = \{x \in U : N_1^C(x) \cap A \neq \emptyset\} = \bar{A}, \tag{13}$$

$$r^-(A) = \{x \in U : \overline{\{x\}} \subseteq A\}, \quad r^+(A) = \{x \in U : \overline{\{x\}} \cap A \neq \emptyset\} = \bigcup \{N_1^C(x) : x \in A\}, \tag{14}$$

$$s^-(A) = \{x \in U : N_1^C(x) \subseteq A \vee \overline{\{x\}} \subseteq A\}, \quad s^+(A) = \{x \in U : N_1^C(x) \cap A \neq \emptyset \wedge \overline{\{x\}} \cap A \neq \emptyset\}, \tag{15}$$

$$b^-(A) = \{x \in U : N_1^C(x) \cup \overline{\{x\}} \subseteq A\}, \quad b^+(A) = \{x \in U : (N_1^C(x) \cup \overline{\{x\}}) \cap A \neq \emptyset\}, \tag{16}$$

$$z^-(A) = \{x \in U : \overline{N_1^C(x)} \subseteq A\}, \quad z^+(A) = \{x \in U : \overline{N_1^C(x)} \cap A \neq \emptyset\}, \tag{17}$$

$$\text{COM}^-(A) = \bigcup \{B \in U/\sim : B \subseteq A\}, \quad \text{COM}^+(A) = \bigcup \{B \in U/\sim : B \cap A \neq \emptyset\}. \tag{18}$$

Apart from the above operators, Zhao also considered the following dual pair of approximation operators due to Samanta and Chakraborty [8,9]:

Definition 14. [8,9] Let (U, \mathbb{C}) be a covering approximation space. The lower and upper approximation operators \underline{P}_4 and \bar{P}_4 are defined by, for $A \subseteq U$,

$$\underline{P}_4(A) = \bigcup \{P_x^C : x \in U \wedge P_x^C \subseteq A\}, \quad \bar{P}_4(A) = \bigcup \{P_x^C : x \in U \wedge P_x^C \cap A \neq \emptyset\}. \tag{19}$$

where the adhesion P_x^C of x in U is defined by

$$P_x^C = \{y \in U : (\forall K \in \mathbb{C})(x \in K \Leftrightarrow y \in K)\} = \{y \in U : N_1^C(x) = N_1^C(y)\}. \tag{20}$$

Note that $\{P_x^C : x \in U\}$ is a partition of the universe U . Zhao [14] examined the partial order relations with respect to \leq between the approximation operators in Definitions 13 and 14: for the upper approximation operators it was derived in [14] that

$$\bar{P}_4 \leq s^+ \leq I^+ \leq b^+ \leq z^+ \leq \text{COM}^+, \tag{21}$$

$$\bar{P}_4 \leq s^+ \leq r^+ \leq b^+ \leq z^+ \leq \text{COM}^+. \tag{22}$$

In general, these partial order relations are strict. The upper approximation operators I^+ and r^+ are incomparable with respect to \leq .

2. Relationship of the topological approximation operators with the general framework

In this section, we relate the approximation operators from Section 1.3 to those from Section 1.2. Throughout the section, we assume that (U, \mathbb{C}) is a covering approximation space. First, in Section 2.1, we will show that all of the operators from Section 1.3, except for s^- and s^+ , are element-based approximation operators. In Section 2.2, we will discuss partial order relations between different neighborhood operators. Finally, in Section 2.3, we will embed the topological approximation operators into the Hasse diagram discussed in [2].

2.1. Neighborhood operators corresponding to topological approximation operators

First, we discuss that the pairs (I^-, I^+) and (r^-, r^+) are pairs of element-based approximation operators.

Proposition 3. Let (U, \mathbb{C}) be a covering approximation space, then $(I^-, I^+) = (\text{apr}_{N_1^C}, \overline{\text{apr}}_{N_1^C})$.

Proof. This follows directly from the definition. \square

As a result, by Definition 3, the neighborhood operator N_1^C effectively generates the interior and closure operator of the topology induced by \mathbb{C} . Moreover, as the following proposition shows, the inverse neighborhood operator $(N_1^C)^{-1}$ also has a topological interpretation:

Proposition 4. Let (U, \mathbb{C}) be a covering approximation space, \mathcal{T} the topology induced by \mathbb{C} and $x \in U$. Then

$$\overline{\{x\}} = (N_1^C)^{-1}(x).$$

Proof. Assume $x \in U$. From [14] we obtain that the closure of $\{x\}$ with respect to the induced topology is given by $U \setminus \left(\bigcup_{K \in \mathbb{C}, x \notin K} K \right)$. Hence, we find

$$\begin{aligned} \overline{x} &= U \setminus \left(\bigcup_{K \in \mathbb{C}, x \notin K} K \right) = \left\{ y \in U : y \notin \bigcup_{K \in \mathbb{C}, x \notin K} K \right\} \\ &= \{y \in U : (\forall K \in \mathbb{C})(x \notin K \Rightarrow y \notin K)\} \\ &= \{y \in U : (\forall K \in \mathbb{C})(y \in K \Rightarrow x \in K)\} \\ &= \left\{ y \in U : x \in N_1^{\mathbb{C}}(y) \right\} = (N_1^{\mathbb{C}})^{-1}(x). \quad \square \end{aligned}$$

Corollary 2. Let (U, \mathbb{C}) be a covering approximation space, then $(r^-, r^+) = (\underline{\text{apr}}_{(N_1^{\mathbb{C}})^{-1}}, \overline{\text{apr}}_{(N_1^{\mathbb{C}})^{-1}})$.

Continuing, it is straightforward to check that the dual pairs (b^-, b^+) , (z^-, z^+) , $(\text{COM}^-, \text{COM}^+)$ and (P_4, \overline{P}_4) are also element-based approximation operators. We will denote the corresponding neighborhood operators by $N_b^{\mathbb{C}}$, $N_z^{\mathbb{C}}$, $N_{\text{COM}}^{\mathbb{C}}$ and $N_{P_4}^{\mathbb{C}}$, respectively, i.e., for x in U ,

$$N_b^{\mathbb{C}}(x) = N_1^{\mathbb{C}}(x) \cup \overline{x}, \tag{23}$$

$$N_z^{\mathbb{C}}(x) = \overline{N_1^{\mathbb{C}}(x)}, \tag{24}$$

$$N_{\text{COM}}^{\mathbb{C}}(x) = [x]_{\sim}, \tag{25}$$

$$N_{P_4}^{\mathbb{C}}(x) = P_x^{\mathbb{C}}. \tag{26}$$

All of the operators are reflexive and symmetric, so they coincide with their inverse neighborhood operators. Moreover, $N_{\text{COM}}^{\mathbb{C}}$ and $N_{P_4}^{\mathbb{C}}$ are equivalence relations and thus are also transitive. Neither $N_b^{\mathbb{C}}$ nor $N_z^{\mathbb{C}}$ are transitive, as the following example demonstrates.

Example 1. Consider the covering approximation space (U, \mathbb{C}) with $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$, then the induced topology is given by

$$\mathcal{T} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{2, 3, 4\}, U\}.$$

We find, for example, that $N_b^{\mathbb{C}}(2) = N_z^{\mathbb{C}}(2) = \{2, 4\}$ and $N_b^{\mathbb{C}}(4) = N_z^{\mathbb{C}}(4) = \{2, 3, 4\}$. So, $4 \in N_b^{\mathbb{C}}(2) = N_z^{\mathbb{C}}(2)$, but $N_b^{\mathbb{C}}(4) = N_z^{\mathbb{C}}(4) \not\subseteq N_b^{\mathbb{C}}(2) = N_z^{\mathbb{C}}(2)$. Hence, $N_b^{\mathbb{C}}$ and $N_z^{\mathbb{C}}$ are not transitive.

To conclude this section, the following example shows that (s^-, s^+) does not meet the criteria of an element-based dual pair of approximation operators.

Example 2. Let $U = \{1, 2, 3\}$ and $\mathbb{C} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$, then the induced topology is given by

$$\mathcal{T} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}\}.$$

It holds that $N_1^{\mathbb{C}}(1) = \{1\}$, $N_1^{\mathbb{C}}(2) = \{1, 2\}$, $N_1^{\mathbb{C}}(3) = \{1, 2, 3\}$, and $\overline{1} = \{1, 2, 3\}$, $\overline{2} = \{2, 3\}$, $\overline{3} = \{3\}$. It holds that $s^+(\{1\}) = \{1\}$ and $s^+(\{3\}) = \{3\}$, yet $s^+(\{1, 3\}) = \{1, 2, 3\}$. Therefore, by Proposition 2, s^+ is not an element-based approximation operator.

2.2. Relationships between neighborhood operators

Next, we will study how the neighborhood operators corresponding to topological rough set approximation operators fit in with the general framework described in Section 1.1. To this aim, we want to discuss partial order relations with respect to the pre-order \preceq . For the 22 neighborhood operators A–M presented in Table 2 the partial order relations were discussed in [2] and the Hasse diagram for these 22 neighborhood operators with respect to \preceq was presented in that paper. We want to add the four neighborhood operators discussed in Section 2.1 to this Hasse diagram. First, consider the following inclusion relationships:

Proposition 5. Let (U, \mathbb{C}) be a covering approximation space, then

$$N_{P_4}^{\mathbb{C}} \preceq N_1^{\mathbb{C}} \preceq N_b^{\mathbb{C}} \preceq N_z^{\mathbb{C}} \preceq N_{\text{COM}}^{\mathbb{C}}. \tag{27}$$

$$N_{P_4}^{\mathbb{C}} \preceq (N_1^{\mathbb{C}})^{-1} \preceq N_b^{\mathbb{C}} \preceq N_z^{\mathbb{C}} \preceq N_{\text{COM}}^{\mathbb{C}}. \tag{28}$$

Proof. Immediately from Proposition 1 and Equations (21) and (22). \square

We also have the following essential inclusion relationships which hold between instances of the two collections of neighborhood operators.

Proposition 6. Let (U, \mathbb{C}) be a covering approximation space, then

- (a) $N_b^{\mathbb{C}} \leq N_4^{\mathbb{C}_3}$,
- (b) $N_4^{\mathbb{C}_3} \leq N_{\text{COM}}^{\mathbb{C}}$,
- (c) $N_z^{\mathbb{C}} \leq N_4^{\mathbb{C}_4}$.

Proof. Let $x \in U$ and \mathcal{T} the induced topology by \mathbb{C} .

- (a) Let $y \in U$ with $y \in N_b^{\mathbb{C}}(x)$, then either $y \in N_1^{\mathbb{C}}(x)$ or $y \in (N_1^{\mathbb{C}})^{-1}(x)$. In both cases it holds that $y \in N_4^{\mathbb{C}_3}(x)$.
- (b) Let $y \in U$ with $y \in N_4^{\mathbb{C}_3}(x)$, then there exists $K \in \mathbb{C}_3$ such that $x, y \in K$. In other words, there exists $z \in U$ such that $x, y \in N_1^{\mathbb{C}}(z)$. Consider the topology \mathcal{T} induced by \mathbb{C} and $V = \{x, y, z\}$. It is clear that the topological subspace (V, \mathcal{T}_V) of (U, \mathcal{T}) is connected, since $N_1^{\mathbb{C}}(z) \cap V = \{x, y, z\}$ is the smallest open set containing the element z . As a consequence, $[x]_{\sim} = [y]_{\sim}$ and hence $y \in N_{\text{COM}}^{\mathbb{C}}(x)$.
- (c) We shall prove that $\text{co}(N_4^{\mathbb{C}_4}(x)) \subseteq \text{co}(N_z^{\mathbb{C}}(x))$. Assume $y \in \text{co}(N_4^{\mathbb{C}_4}(x))$ and denote $\mathcal{T}^{\text{co}} = \{\text{co} X : X \in \mathcal{T}\}$. We want to prove that $y \in \text{co}(N_z^{\mathbb{C}}(x))$, i.e.,

$$\begin{aligned} y \in \text{co}(N_z^{\mathbb{C}}(x)) &\Leftrightarrow y \notin \bigcap \{Y \in \mathcal{T}^{\text{co}} : N_1^{\mathbb{C}}(x) \subseteq Y\} \\ &\Leftrightarrow y \notin \bigcap \{\text{co} X : X \in \mathcal{T}, N_1^{\mathbb{C}}(x) \subseteq \text{co} X\} \\ &\Leftrightarrow y \notin \bigcap \{\text{co} X : X \in \mathcal{T}, N_1^{\mathbb{C}}(x) \cap X = \emptyset\} \\ &\Leftrightarrow y \in \bigcup \{X \in \mathcal{T} : N_1^{\mathbb{C}}(x) \cap X = \emptyset\}. \end{aligned}$$

By definition, it holds that $N_1^{\mathbb{C}}(y) \in \mathcal{T}$ and $y \in N_1^{\mathbb{C}}(y)$. Since $y \in \text{co}(N_4^{\mathbb{C}_4}(x))$, it holds that $N_4^{\mathbb{C}_4}(x) \cap N_1^{\mathbb{C}}(y) = \emptyset$, hence, $N_1^{\mathbb{C}}(x) \cap N_1^{\mathbb{C}}(y) = \emptyset$. Therefore, $y \in \bigcup \{X \in \mathcal{T} : N_1^{\mathbb{C}}(x) \cap X = \emptyset\}$ and thus, $y \in \text{co}(N_z^{\mathbb{C}}(x))$. \square

The only other partial order relations which hold with respect to \leq follow from the transitivity of \leq . The following example illustrates that there exist no other partial order relationships between the two collections of neighborhood operators.

Example 3. The following four covering approximation spaces provide all the counterexamples for the partial order relations which do not hold:

- $(U, \mathbb{C})_1$ with $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1\}, \{3\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}\}$ for $x = 1, 2, 3, 4$, with the induced topology \mathcal{T} given by

$$\mathcal{T} = \{\emptyset, \{1\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, U\}.$$

- $(U, \mathbb{C})_2$ with $U = \{1, 2, 3, 4, 5\}$ and $\mathbb{C} = \{\{1, 2\}, \{2, 3, 4\}, \{4, 5\}\}$ for $x = 1, 2$, with the induced topology \mathcal{T} given by

$$\mathcal{T} = \{\emptyset, \{2\}, \{4\}, \{1, 2\}, \{4, 5\}, \{1, 2, 4\}, \{2, 3, 4\}, \{2, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{2, 3, 4, 5\}, U\}.$$

- $(U, \mathbb{C})_3$ with $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$ for $x = 2$, with the induced topology \mathcal{T} given by

$$\mathcal{T} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{2, 3, 4\}, U\}.$$

- $(U, \mathbb{C})_4$ with $U = \{1, 2, 3, 4\}$ and $\mathbb{C} = \{\{1\}, \{3\}, \{4\}, \{2, 3, 4\}\}$ for $x = 2, 3$, with the induced topology \mathcal{T} given by

$$\mathcal{T} = \{\emptyset, \{1\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, U\}.$$

For example, to see that the partial order relation $Q \leq M$ does not hold, consider $(U, \mathbb{C})_2$, then $N_4^{\mathbb{C}_4}(1) = \{1, 2, 3, 4\}$ and $N_{\text{COM}}^{\mathbb{C}}(1) = U$.

In Fig. 1, we show the corresponding Hasse diagram with respect to \leq for the 26 neighborhood operators $A - Q$ presented in Table 2, highlighting the position of the neighborhood operators obtained from the framework of Zhao [14].

To end this section, we prove the following lemma and proposition:

Lemma 1. Let (U, \mathbb{C}) be a covering approximation space and \mathcal{T} the topology induced by \mathbb{C} . Let N be a reflexive neighborhood operator such that $N \leq N_{\text{COM}}^{\mathbb{C}}$, then

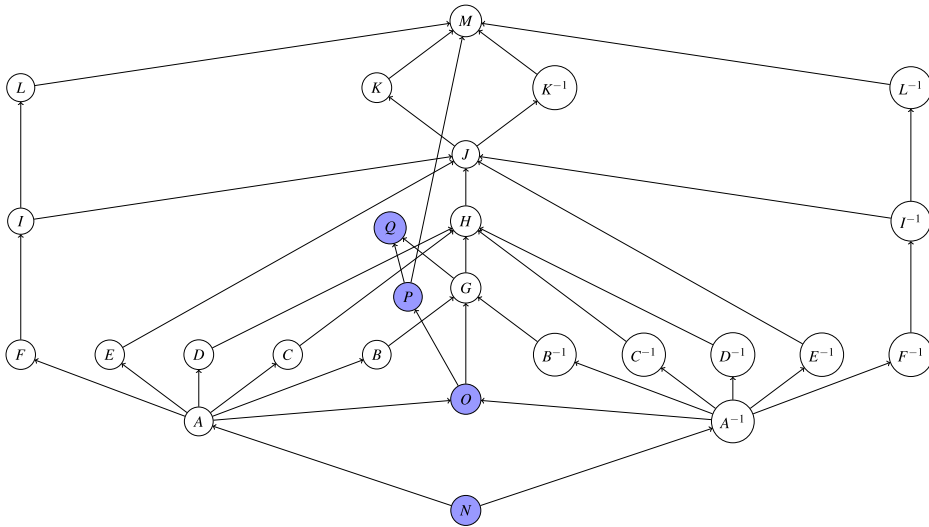


Fig. 1. Hasse diagram for the neighborhood operators in Table 2, where we have highlighted the four neighborhood operators derived from [14].

$$\forall x \in U: N_{COM}^C(x) = \bigcup \{N(z) : z \in N_{COM}^C(x)\}.$$

Proof. As N is reflexive, we have for $x \in U$ that $N_{COM}^C(x) = \bigcup_{z \in N_{COM}^C(x)} \{z\} \subseteq \bigcup_{z \in N_{COM}^C(x)} N(z)$. On the other hand, $N(z) \subseteq N_{COM}^C(z) = N_{COM}^C(x)$, for all $z \in N_{COM}^C(x)$. Hence, $\bigcup_{z \in N_{COM}^C(x)} N(z) \subseteq N_{COM}^C(x)$. \square

Proposition 7. Let (U, \mathbb{C}) be a covering approximation space and \mathcal{T} the topology induced by \mathbb{C} . Let N be a reflexive neighborhood operator such that $N \preceq N_{COM}^C$ and $\{N(x) : x \in U\} \subseteq \mathcal{T}$, i.e., every neighborhood $N(x)$ is open. If $\{N(x) : x \in U\}$ is a partition of U , it holds that $N = N_{COM}^C$.

Proof. Let $x \in U$, by Lemma 1 it holds that

$$N_{COM}^C(x) = \bigcup_{z \in N_{COM}^C(x)} N(z) = \left(\bigcup_{z \in N(x)} N(z) \right) \cup \left(\bigcup_{z \in N_{COM}^C(x) \setminus N(x)} N(z) \right) =: A \cup B.$$

We have that A and B are open, disjoint sets such that their union is $N_{COM}^C(x)$. As $A \neq \emptyset$ and $N_{COM}^C(x)$ is connected, we have that $B = \emptyset$. Hence, we conclude that $N_{COM}^C(x) = N(x)$ for all $x \in U$. \square

Remark 1. The reflexive neighborhood operators which satisfy the condition $N \preceq N_{COM}^C$ are the neighborhood operators A, A^{-1} , B, B^{-1} , G, N, O, P and Q. When the universe U is finite, the operators A, B, G and Q also satisfy the condition $\{N(x) : x \in U\} \subseteq \mathcal{T}$.

2.3. Partial order relationships between topological approximation operators and general framework

In this section, we discuss the partial order relations with respect to \leq between the approximation operators of the general framework of [2,3] and the topological approximation operators discussed in [14], all listed in Table 2. By Equation (6) it is sufficient to study the partial order relationships between the upper approximation operators. By Proposition 1, the partial order relations between the element-based approximation operators given by the operators 1 – 26 are immediately obtained from the results represented in Fig. 1. Furthermore, we have the following proposition:

Proposition 8. Let (U, \mathbb{C}) be a covering approximation space, then

$$s^+ \leq \overline{\text{apr}}_{N_1^C} \tag{29}$$

$$s^+ \leq \overline{\text{apr}}_{(N_1^C)^{-1}} \tag{30}$$

Table 1
Incomparabilities between approximation operators based on the covering approximation space $(U, \mathbb{C})_1$.

A	$s^+(A)$	$\bar{P}_4(A)$	$b^+(A)$	$\overline{\text{apr}}'_{\mathbb{C}}(A)$	$\overline{\text{apr}}'_{\mathbb{C}_n}(A)$	$\overline{\text{apr}}'_{\mathbb{C}_2}(A)$	$\overline{\text{apr}}'_{\mathbb{C}_4}(A)$	$\overline{\text{apr}}'_{S_n}(A)$	$\overline{\text{apr}}^z_{\mathbb{C}}(A)$	$\overline{\text{apr}}^{\text{zhu}}_{\mathbb{C}}(A)$
{2}	{2}	{2}	{2, 4}	{2}	{2}	{2}	{2}	{2, 4}	{2, 4}	
{4}	{4}	{4}	{2, 4}	{2, 4}	{2, 4}	{2, 4}	{2, 4}	{4}	{4}	{1, 2, 3, 4}
{1, 2}			{1, 2, 4}					{1, 2, 3, 4}		{1, 2, 3, 4}
{3, 4}			{2, 3, 4}		{1, 2, 3, 4}	{1, 2, 3, 4}	{1, 2, 3, 4}		{3, 4}	{3, 4}

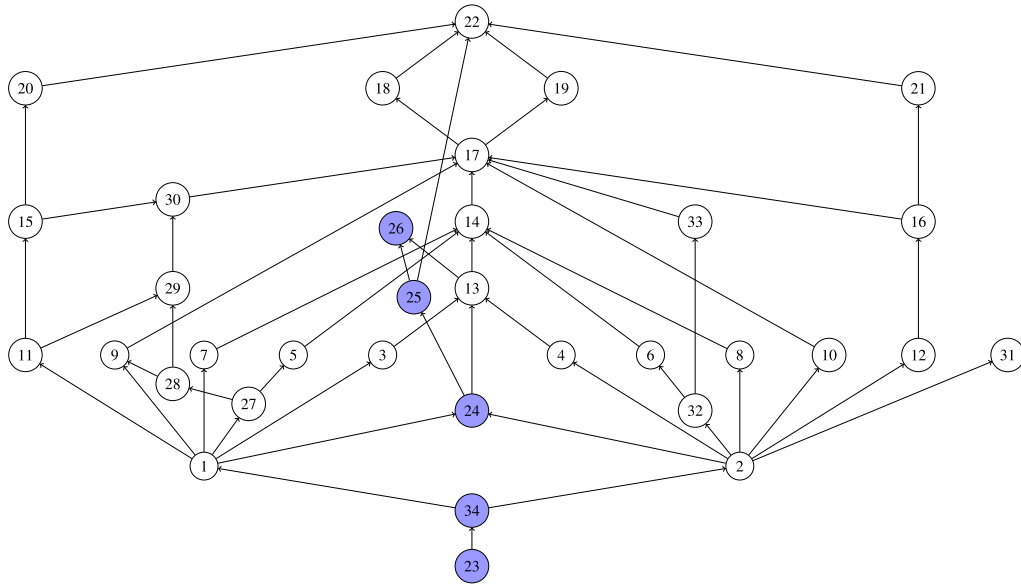


Fig. 2. Hasse diagram of upper approximation operators from Table 2, where we have highlighted the approximation operators from [14].

Proof. Direct from the definition and Proposition 4. □

As illustrated in the next example, there are no other partial order relations between the approximation operators of the general framework and the topological approximation operators.

Example 4. Consider the approximation space $(U, \mathbb{C})_1$ from Example 3. Note that $b^+(A) = z^+(A) = \text{COM}^+(A)$ for each $A \subseteq U$. In Table 1 we provide counterexamples for partial order relations with respect to \leq .

Given Table 1, we only need to discuss whether $\overline{\text{apr}}'_{\mathbb{C}}$ is smaller than b^+ , z^+ and COM^+ . Consider the approximation space $(U, \mathbb{C})_3$ from Example 3 and let $A = \{2, 3, 4\}$. We have that $b^+(A) = z^+(A) = \text{COM}^+(A) = A$, while $\overline{\text{apr}}'_{\mathbb{C}}(A) = \{1, 2, 3, 4\}$.

The Hasse diagram of the partial order relationships between the upper approximation operators of Table 2 is presented in Fig. 2. The five topological approximation operators which are added to the general framework are highlighted.

3. Conclusion and future work

In this note, we have incorporated the results on topological approximation operators presented in [14] in the general framework provided in [2]. Of the seven dual pairs of topological approximation operators, two pairs coincide with pairs of the general framework. Of the other five pairs, four of them are element-based approximation operators. We have studied how their respective neighborhood operators fit in the framework of neighborhood operators described in [3]. Moreover, we have added the five pairs of topological approximation operators to our general framework of covering-based rough set approximation operators.

The Hasse diagram of upper approximation operators also sheds light on the application perspective of covering-based rough sets, since smaller upper approximation operators yield a higher accuracy. Therefore we may conclude that not only element-based approximation operators are useful for applications such as feature selection. However, little research has been done on the application of non-neighborhood-based rough set models in machine learning, so we consider this as an interesting direction for future research.

Table 2

Overview of covering-based rough set upper approximation operators. For element-based approximation operators, we also indicate the corresponding neighborhood operator in the column with header "N".

No.	Upper Appr.	N
1	$\overline{\text{apr}}_{A}, \overline{\text{apr}}'_{C_3}, I^+$	$A : N_1^C, N_2^{C_1}, N_1^{C_3}, N_1^{C_n}, N_2^{C_3}$
2	$\overline{\text{apr}}_{A^{-1}}, r^+$	$A^{-1} : (N_1^C)^{-1}, (N_1^{C_1})^{-1}, (N_1^{C_3})^{-1}, (N_1^{C_n})^{-1}, (N_2^{C_3})^{-1}$
3	$\overline{\text{apr}}_B$	$B : N_3^{C_3}$
4	$\overline{\text{apr}}_{B^{-1}}$	$B^{-1} : (N_3^{C_3})^{-1}$
5	$\overline{\text{apr}}_C$	$C : N_2^C, N_2^{C_1}$
6	$\overline{\text{apr}}_{C^{-1}}$	$C^{-1} : (N_2^C)^{-1}, (N_2^{C_1})^{-1}$
7	$\overline{\text{apr}}_D$	$D : N_3^{C_1}$
8	$\overline{\text{apr}}_{D^{-1}}$	$D^{-1} : (N_3^{C_1})^{-1}$
9	$\overline{\text{apr}}_E$	$E : N_2^{C_n}$
10	$\overline{\text{apr}}_{E^{-1}}$	$E^{-1} : (N_2^{C_n})^{-1}$
11	$\overline{\text{apr}}_F$	$F : N_3^C, N_1^{C_2}, N_3^{C_2}, N_3^{C_n}$
12	$\overline{\text{apr}}_{F^{-1}}$	$F^{-1} : (N_3^C)^{-1}, (N_1^{C_2})^{-1}, (N_3^{C_2})^{-1}, (N_3^{C_n})^{-1}$
13	$\overline{\text{apr}}_G$	$G : N_4^{C_3}, (N_4^{C_3})^{-1}$
14	$\overline{\text{apr}}_H$	$H : N_4^{C_1}, (N_4^{C_1})^{-1}$
15	$\overline{\text{apr}}_I$	$I : N_1^{C_4}$
16	$\overline{\text{apr}}_{I^{-1}}$	$I^{-1} : (N_1^{C_4})^{-1}$
17	$\overline{\text{apr}}_J$	$J : N_4^C, N_2^{C_2}, N_4^{C_2}, N_4^{C_n}, (N_4^C)^{-1}, (N_2^{C_2})^{-1}, (N_4^{C_2})^{-1}, (N_4^{C_n})^{-1}$
18	$\overline{\text{apr}}_K$	$K : N_2^{C_4}$
19	$\overline{\text{apr}}_{K^{-1}}$	$K^{-1} : (N_2^{C_4})^{-1}$
20	$\overline{\text{apr}}_L$	$L : N_3^{C_4}$
21	$\overline{\text{apr}}_{L^{-1}}$	$L^{-1} : (N_3^{C_4})^{-1}$
22	$\overline{\text{apr}}_M$	$M : N_4^{C_4}, (N_4^{C_4})^{-1}$
23	$\overline{\text{apr}}_{N}, \overline{P}_4$	$N : N_{P_4}^C, (N_{P_4}^C)^{-1}$
24	$\overline{\text{apr}}_O, b^+$	$O : N_b^C, (N_b^C)^{-1}$
25	$\overline{\text{apr}}_P, z^+$	$P : N_z^C, (N_z^C)^{-1}$
26	$\overline{\text{apr}}_Q, \text{COM}^+$	$Q : N_{\text{COM}}^C, (N_{\text{COM}}^C)^{-1}$
27	$\overline{\text{apr}}'_{C_1}, \overline{\text{apr}}'_{C_1}, \overline{\text{apr}}_{S_u}$	
28	$\overline{\text{apr}}'_{C_n}$	
29	$\overline{\text{apr}}'_{C_2}$	
30	$\overline{\text{apr}}'_{C_4}$	
31	$\overline{\text{apr}}_{S_n}$	
32	$\overline{\text{apr}}_C^z$	
33	$\overline{\text{apr}}_C^{\text{zhu}}$	
34	s^+	

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